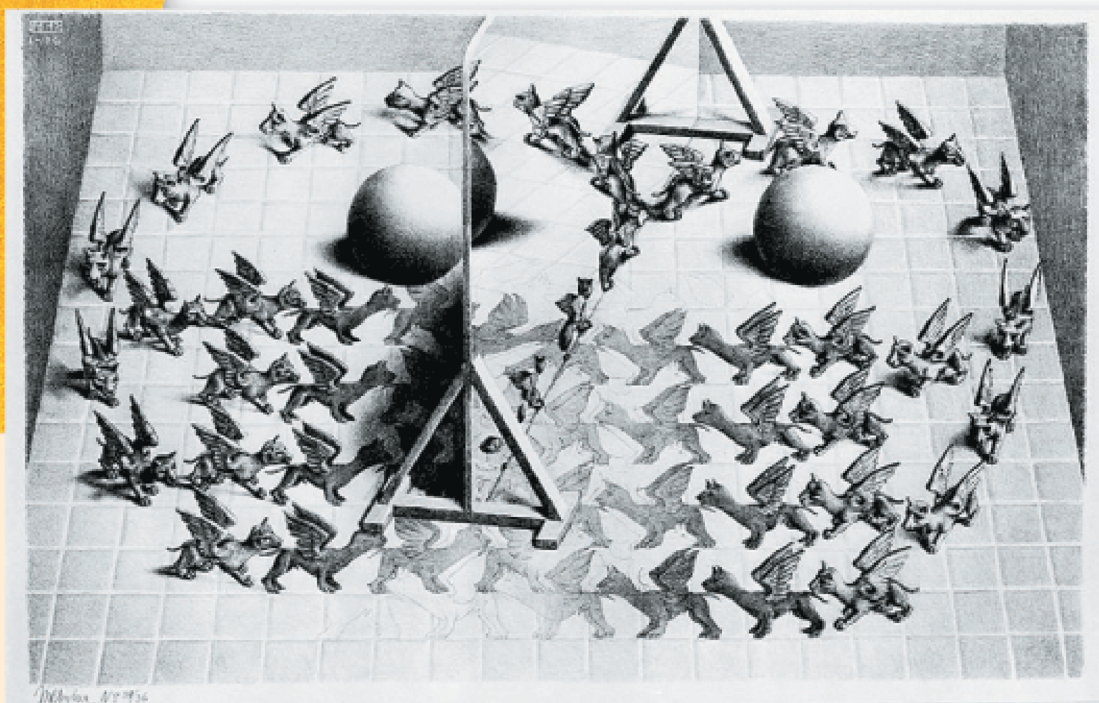
CHAPTER 7

Transformations and Tessellations



I believe that producing pictures, as I do, is almost solely a question of wanting so very much to do it well.

M. C. ESCHER

Magic Mirror, M. C. Escher, 1946

OBJECTIVES

In this chapter you will

- discover some basic properties of transformations and symmetry
- learn more about symmetry in art and nature
- create tessellations

LESSON 7.1

Transformations and Symmetry

Symmetry is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.

HERMANN WEYL

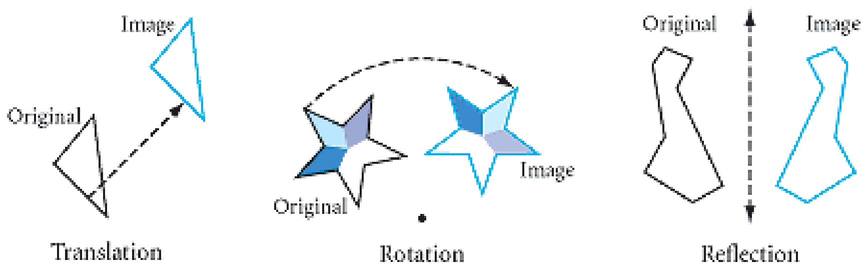
Each light bulb is an image of every other light bulb.

By moving all the points of a geometric figure according to certain rules, you can create an **image** of the original figure. This process is called **transformation.** Each point on the original figure corresponds to a point on its image. The image of point A after a transformation of any type is called point A' (read "A prime") , as shown in the transformation of $\triangle ABC$ to $\triangle A'B'C'$ on the facing page.

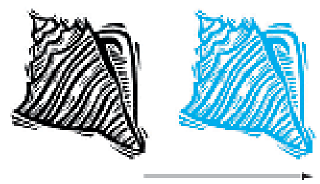
Frieze of bowmen from the Palace of Artaxerxes II in Susa, Iran

If the image is congruent to the original figure, the process is called **rigid transformation**, or **isometry.** A transformation that does not preserve the size and shape is called or **nonrigid transformation.** For example, if an image is reduced or enlarged, or if the shape changes, its transformation is nonrigid.

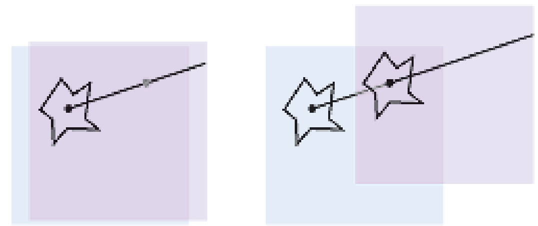
Three types of rigid transformation are translation, rotation, and reflection. You have been doing translations, rotations, and reflections in your patty-paper investigations and in exercises on the coordinate plane, using (x, y) rules.



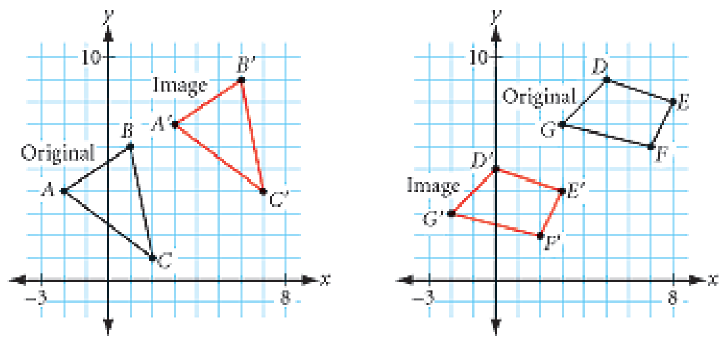
Translation is the simplest type of isometry. You can model a translation by tracing a figure onto patty paper, then sliding it along a straight path without turning it. Notice that when you slide the figure, all points move the same distance along parallel paths to form its image. That is, each point in the image is equidistant from the point that corresponds to it in the original figure. This distance, because it is the same for all points, is called the distance of the translation. A translation also has a particular direction. So you can use a **translation vector** to describe the translation.



Translation vector

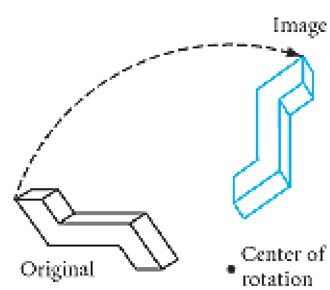


Translating with patty paper

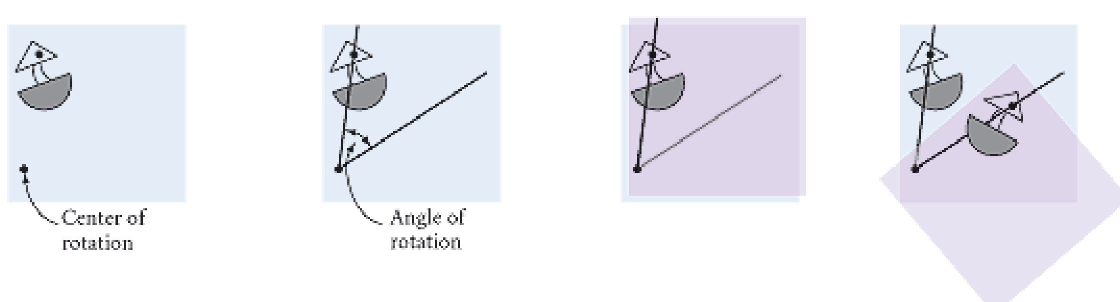


Translations on a coordinate grid

Rotation is another type of isometry. In a rotation, all the points in the original figure rotate, or turn, an identical number of degrees about a fixed center point. You can define a rotation by its center point, the number of degrees it's turned, and whether it's turned clockwise or counterclockwise. If no direction is given, assume the direction of rotation is counterclockwise.

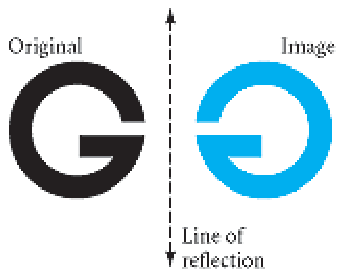


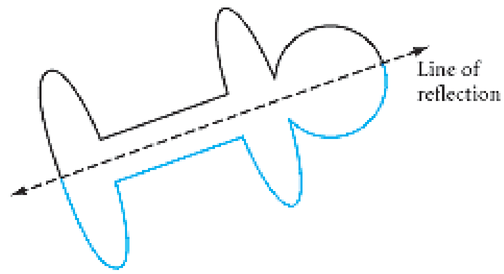
You can model a rotation by tracing over a figure, then putting your pencil point on a point on the patty paper and rotating the patty paper about the point.

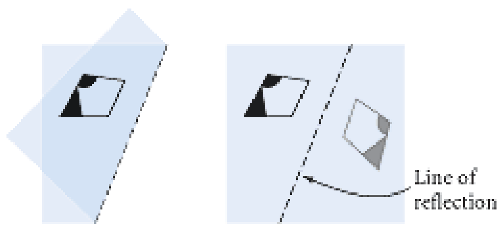


Rotating with patty paper

Reflection is a type of isometry that produces a figure's "mirror image." If you draw a figure onto a piece of paper, place the edge of a mirror perpendicular to your paper, and look at the figure in the mirror, you will see the reflected image of the figure. The line where the mirror is placed is called the **line of reflection.**







Reflecting with patty paper

History CONNECTION

Leonardo da Vinci (1452–1519, Italy) wrote his scientific discoveries backward so that others couldn't read them and punish him for his research and ideas. His knowledge and authority in almost every subject is astonishing even today. Scholars marvel at his many notebooks containing research into anatomy, mathematics, architecture, geology, meteorology, machinery, and botany, as well as his art masterpieces, like the *Mona Lisa*. Notice his plans for a helicopter in the manuscript at right!

Investigation

The Basic Property of a Reflection

You will need

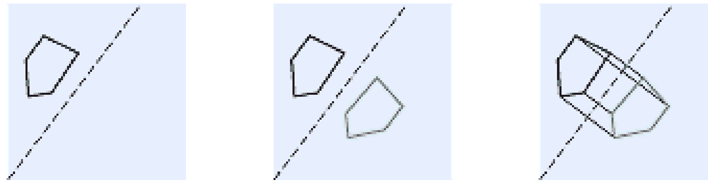
- patty paper
- a straightedge

In this investigation you'll model reflection with patty paper and discover an important property of reflections.

Step 1 Draw a polygon and a line of reflection next to it on a piece of patty paper.

Step 2 Fold your patty paper along the line of reflection and create the reflected image of your polygon by tracing it.

Step 3 Draw segments connecting each vertex with its image point. What do you notice?



Step 1 Step 2 Step 3

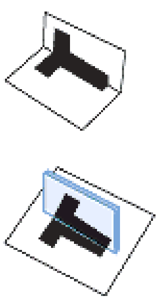
Step 4 Compare your results with those of your group members. Copy and complete the following conjecture.

Reflection Line Conjecture C-67

The line of reflection is the _?_ of every segment joining a point in the original figure with its image.

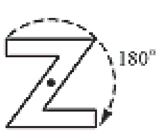
If a figure can be reflected over a line in such a way that the resulting image coincides with the original, then the figure has **reflectional symmetry.** The reflection line is called the **line of symmetry.** The Navajo rug shown below has two lines of symmetry.

The letter *T* has reflectional symmetry. You can test a figure for reflectional symmetry by using a mirror or by folding it.

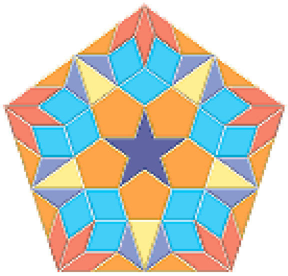


Navajo rug (two lines of symmetry)

The letter *Z* has 2-fold rotational symmetry. When it is rotated 180° and 360° about a center of rotation, the image coincides with the original figure.

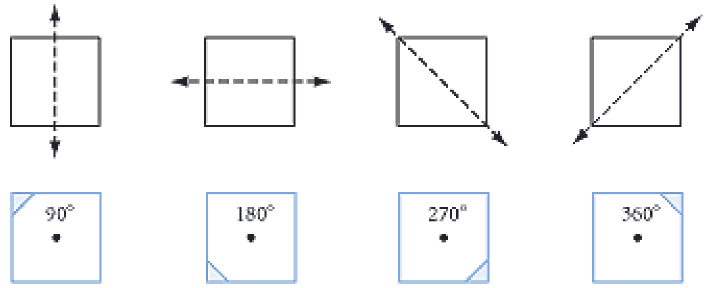


If a figure can be rotated about a point in such a way that its rotated image coincides with the original figure before turning a full 360°, then the figure has **rotational symmetry.** Of course, every image is identical to the original figure after a rotation of any multiple of 360°. However, we don't call a figure symmetric if this is the only kind of symmetry it has. You can trace a figure to test it for rotational symmetry. Place the copy exactly over the original, put your pen or pencil point on the center to hold it down, and rotate the copy. Count the number of times the copy and the original coincide until the copy is back in its original position. Two-fold rotational symmetry is also called **point symmetry.**



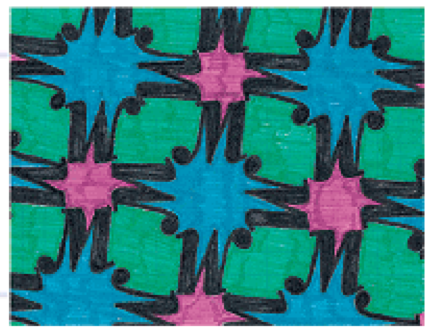
This tile pattern has both 5-fold rotational symmetry and 5-fold reflectional symmetry.

Some polygons have no symmetry or only one kind of symmetry. Regular polygons, however, are symmetric in many ways. A square, for example, has 4-fold reflectional symmetry and 4-fold rotational symmetry.



Art CONNECTION

Reflecting and rotating a letter can produce an unexpected and beautiful design. With the aid of graphics software, a designer can do this quickly and inexpensively. For more information, go to www.keymath.com/DG. This design was created by geometry student Michelle Cotter.



keymath.com Practice Your Skills

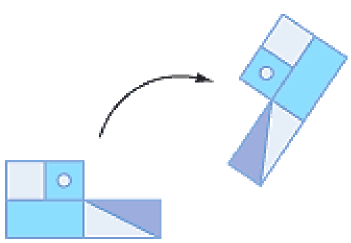
EXERCISES

You will need

Construction tools for Exercises 9 and 10

In Exercises 1–3, say whether the transformations are rigid or nonrigid. Explain how you know.

1.



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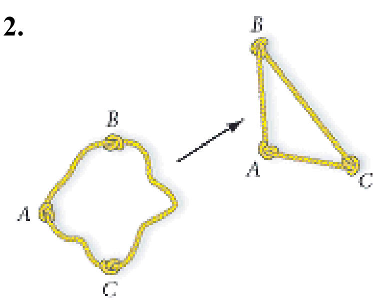

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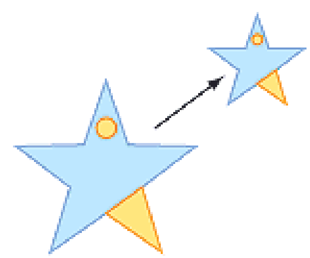
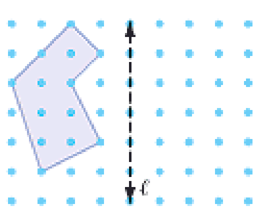
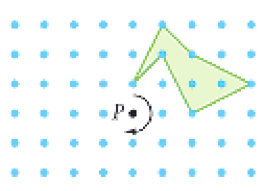


In Exercises 4–6, copy the figure onto graph or square dot paper and perform each transformation.

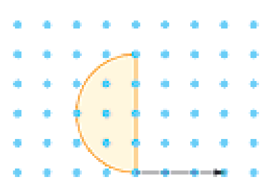
4. Reflect the figure across the line of reflection, line ℓ.



5. Rotate the figure 180° about the center of rotation, point P.



6. Translate the figure by the translation vector.



7. An ice skater gliding in one direction creates several translation transformations. Give another real-world example of translation.

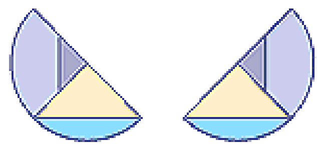
8. An ice skater twirling about a point creates several rotation transformations. Give another real-world example of rotation.

In Exercises 9–11, perform each transformation. Attach your patty paper to your homework.

9. ***Construction*** Use the semicircular figure and its reflected image.

   a. Copy the figure and its reflected image onto a piece of patty paper. Locate the line of reflection. Explain your method.

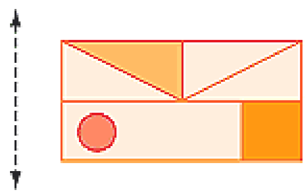
   b. Copy the figure and its reflected image onto a sheet of paper. Locate the line of reflection using a compass and straightedge. Explain your method.



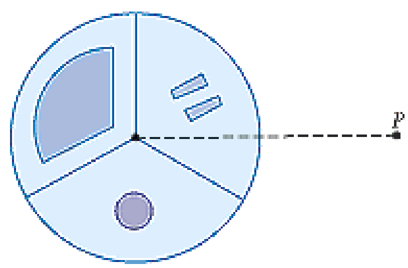
10. ***Construction*** Use the rectangular figure and the reflection line next to it.

   a. Copy the figure and the line of reflection onto a sheet of paper. Use a compass and straightedge to construct the reflected image. Explain your method.

   b. Copy the figure and the line of reflection onto a piece of patty paper. Fold the paper and trace the figure to construct the reflected image.



11. Trace the circular figure and the center of rotation, *P*. Rotate the design 90° clockwise about point *P*. Draw the image of the figure, as well as the dotted line.



In Exercises 12–14, identify the type (or types) of symmetry in each design.

12.

Butterfly

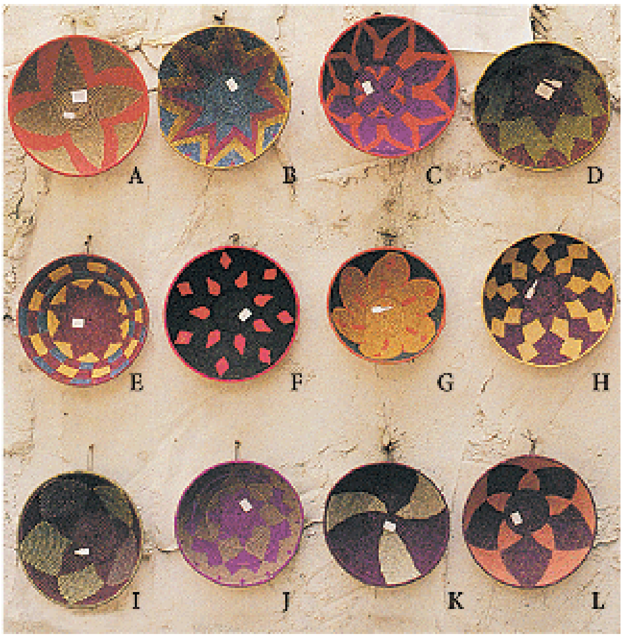
13.

Hmong textile, Laos

14.

The Temple Beth Israel, San Diego's first synagogue, built in 1889

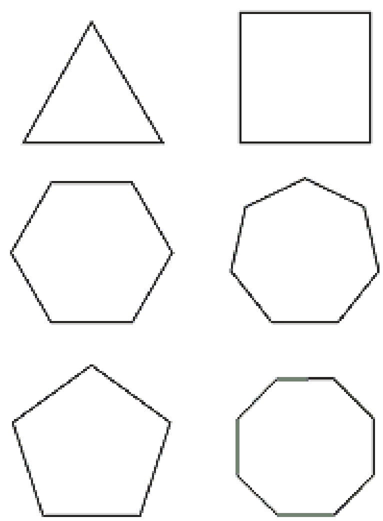
15. All of the woven baskets from Botswana shown below have rotational symmetry and most have reflectional symmetry. Find one that has 7-fold symmetry. Find one with 9-fold symmetry. Which basket has rotational symmetry but not reflectional symmetry? What type of rotational symmetry does it have?



Cultural CONNECTION

For centuries, women in Botswana, a country in southern Africa, have been weaving baskets like the ones you see at left, to carry and store food. Each generation passes on the tradition of weaving choice shoots from the mokola palm and decorating them in beautiful geometric patterns with natural dyes. In the past 40 years, international demand for the baskets by retailers and tourists has given economic stability to hundreds of women and their families.

16. ***Mini-Investigation*** Copy and complete the table below. If necessary, use a mirror to locate the lines of symmetry for each of the regular polygons. To find the number of rotational symmetries, you may wish to trace each regular polygon onto patty paper and rotate it. Then copy and complete the conjecture.



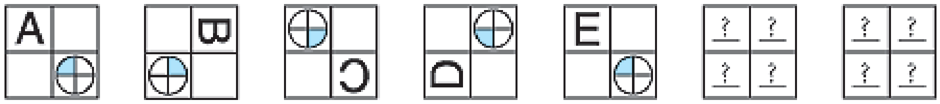
Number of sides of regular polygon	3	4	5	6	7	8	. . .	n
Number of reflectional symmetries		4					. . .	
Number of rotational symmetries ($\leq 360°$)		4					. . .	

A regular polygon of n sides has __?__ reflectional symmetries and __?__ rotational symmetries.

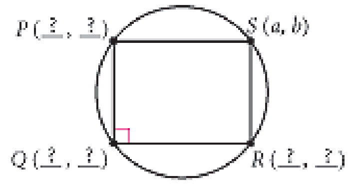
Review

In Exercises 17 and 18, sketch the next two figures.

17. 

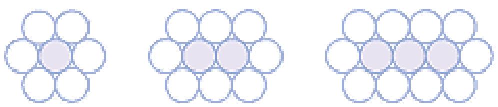
18. 

19. Polygon *PQRS* is a rectangle inscribed in a circle centered at the origin. The slope of $\overline{PS}$ is 0. Find the coordinates of points *P, Q*, and *R* in terms of *a* and *b*.

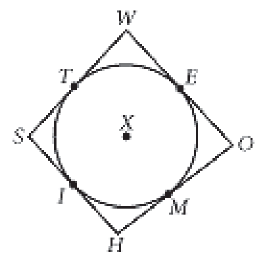


20. Use a circular object to trace a large minor arc. Using either compass-and-straightedge construction or patty-paper construction, locate a point on the arc equally distant from the arc's endpoints. Label it *P*.

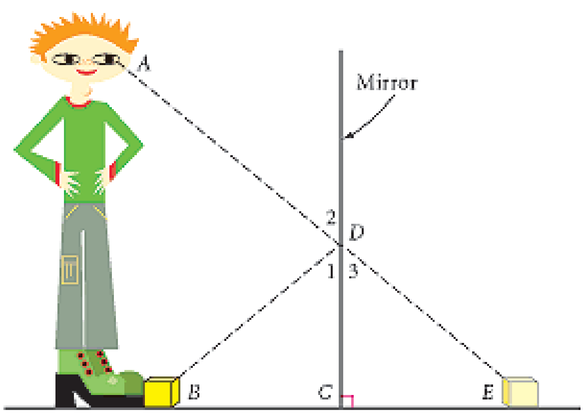
21. If the circle pattern continues, how many total circles (shaded and unshaded) will there be in the 50th figure? How many will there be in the *n*th figure?



22. Quadrilateral *SHOW* is circumscribed about circle *X*. *WO* = 14, *HM* = 4, *SW* = 11, and *ST* = 5. What is the perimeter of *SHOW*?



23. ***Developing Proof*** What you see in a mirror is actually light from an object bouncing off the mirror and traveling to your eye. The object's image seen in the mirror appears as if it were reflected behind the mirror, as far behind the mirror as the object is in front of the mirror. In the diagram at right, assume that the mirror is perpendicular to the ground. Use the fact that the light's incoming angle, $\angle 1$, is congruent to its outgoing angle, $\angle 2$, to explain why *BC* = *CE*.



IMPROVING YOUR ALGEBRA SKILLS

The Difference of Squares

$17^2 - 16^2 = 33$ $\quad$ $25.5^2 - 24.5^2 = 50$ $\quad$ $34^2 - 33^2 = 67$

$58^2 - 57^2 = 115$ $\quad$ $62.1^2 - 61.1^2 = 123.2$ $\quad$ $76^2 - 75^2 = 151$

Can you use algebra to explain why you can just add the two base numbers to get the answer ? (For example, 17 + 16 = 33.)

LESSON 7.2

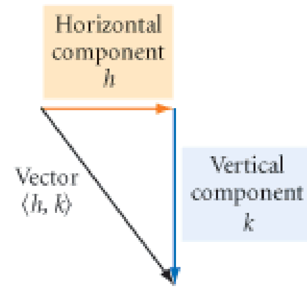
Properties of Isometries

"Why it's a looking-glass book, of course! And, if I hold it up to the glass, the words will all go the right way again."

ALICE IN *THROUGH THE LOOKING-GLASS* BY LEWIS CARROLL

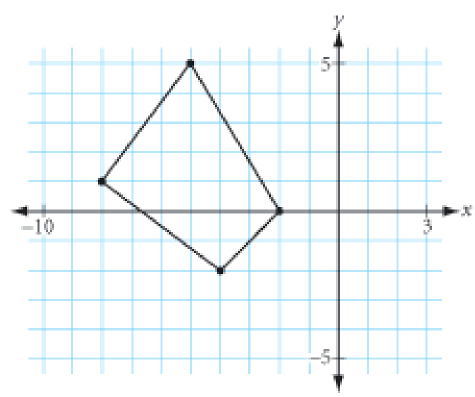
In many earlier exercises, you used **ordered pair rules** to transform polygons on a coordinate plane by relocating their vertices. For any point on a figure, the ordered pair rule $(x, y) \rightarrow (x + h, y + k)$ results in a horizontal move of h units and a vertical move of k units for any numbers h and k. That is, if (x, y) is a point on the original figure, $(x + h, y + k)$ is its corresponding point on the image.

An ordered pair rule can also be written as a vector. As shown at right, a vector is named by its horizontal and vertical components.



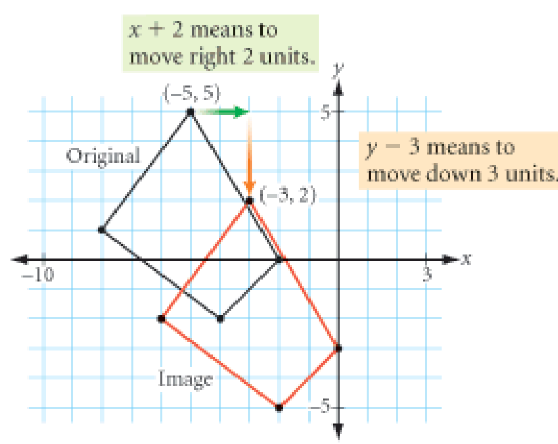
EXAMPLE A

Transform the polygon at right using the rule $(x, y) \rightarrow (x + 2, y - 3)$. Describe the type and direction of the transformation, and write it as a vector.



▶ Solution

Apply the rule to each ordered pair. Every point of the polygon moves right 2 units and down 3 units. This is a translation by the vector $\langle 2, -3\rangle$.



So the ordered pair rule $(x, y) \rightarrow (x + h, y + k)$ results in a translation by the vector $\langle h, k\rangle$, where h is the horizontal component of the vector and k is the vertical component of the vector.

Investigation 1
Transformations on a Coordinate Plane

You will need

- graph paper
- patty paper

In this investigation you will discover (or rediscover) how four ordered pair rules transform a polygon. Each person in your group can choose a different polygon for this investigation.

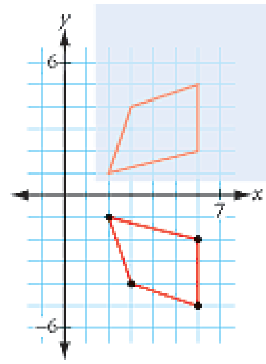
Step 1 On graph paper, create and label four sets of coordinate axes. Draw the same polygon in the same position in a quadrant of each of the four graphs. Write one of these four ordered pair rules below each graph.

a. $(x, y) \rightarrow (-x, y)$

b. $(x, y) \rightarrow (x, -y)$

c. $(x, y) \rightarrow (-x, -y)$

d. $(x, y) \rightarrow (y, x)$



Step 2 Use the ordered pair rule you assigned to each graph to relocate the vertices of your polygon and create its image.

Step 3 Use patty paper to see if your transformation is a reflection, translation, or rotation. Compare your results with those of your group members. Complete the conjecture. Be specific.

Coordinate Transformations Conjecture C-68

The ordered pair rule $(x, y) \rightarrow (-x, y)$ is a __?__.

The ordered pair rule $(x, y) \rightarrow (x, -y)$ is a __?__.

The ordered pair rule $(x, y) \rightarrow (-x, -y)$ is a __?__.

The ordered pair rule $(x, y) \rightarrow (y, x)$ is a __?__.

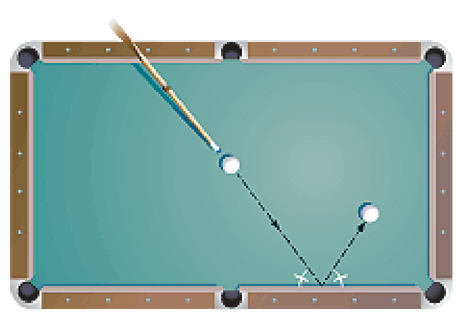
Let's revisit "poolroom geometry." When a ball rolls without spin into a cushion, the outgoing angle is congruent to the incoming angle. This is true because the outgoing and incoming angles are reflections of each other.

Investigation 2
Finding a Minimal Path

You will need

- patty paper
- a protractor

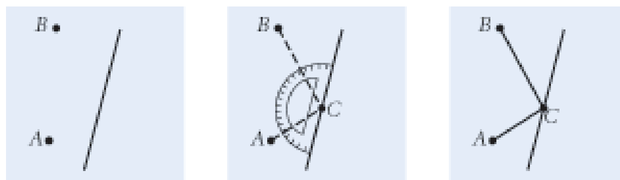
In Chapter 1, you used a protractor to find the path of the ball. In this investigation, you'll discover some other properties of reflections that have many applications in science and engineering. They may even help your pool game!



Step 1 Draw a segment, representing a pool table cushion, near the center of a piece of patty paper. Draw two points, A and B, on one side of the segment.

Step 2 Imagine you want to hit a ball at point A so that it bounces off the cushion and hits another ball at point B. Use your protractor and trial and error to find the point C on the cushion that you should aim for.

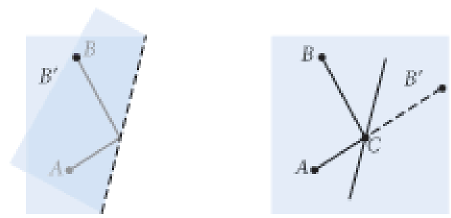
Step 3 Draw $\overline{AC}$ and $\overline{CB}$ to represent the ball's path.



Step 1 Step 2 Step 3

Step 4 Fold your patty paper to draw the reflection of point B across the line segment. Label the image point B'.

Step 5 Unfold the paper and draw a segment from point A to point B'. What do you notice? Does point C lie on segment $\overline{AB'}$? How does the path from A to B' compare to the two-part path from A to C to B?



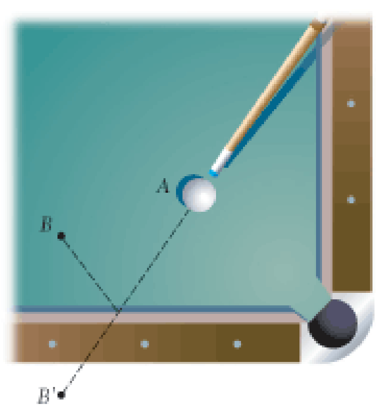
Step 4 Step 5

Step 6 Can you draw any other path from point A to the cushion to point B that is shorter than $AC + CB$? Why or why not? The shortest path from point A to the cushion to point B is called the **minimal path.** Copy and complete the conjecture.

Minimal Path Conjecture C-69

If points A and B are on one side of line ℓ, then the minimal path from point A to line ℓ to point B is found by _?_.

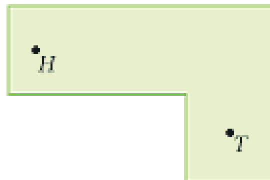
How can this discovery help your pool game? Suppose you need to hit a ball at point A into the cushion so that it will bounce off the cushion and pass through point B. To what point on the cushion should you aim? Visualize point B reflected across the cushion. Then aim directly at the reflected image.



Let's look at a miniature-golf example.

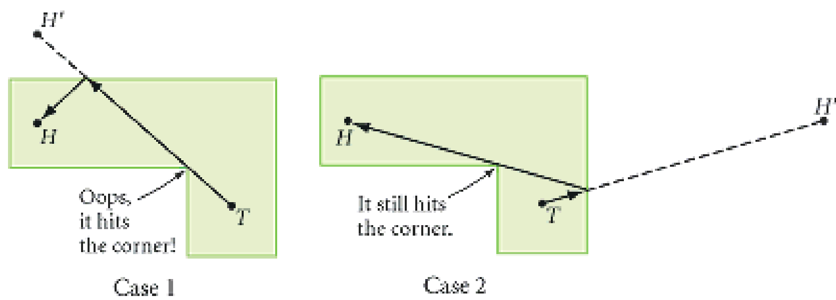
EXAMPLE B

How can you hit the ball at T around the corner and into the hole at H in one shot?

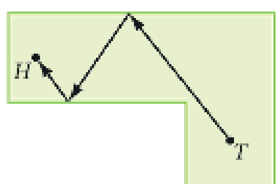


Solution

First, try to get a hole-in-one with a direct shot or with just one bounce off a wall. For one bounce, decide which wall the ball should hit. Visualize the image of the hole across that wall and aim for the reflected hole. There are two possibilities.

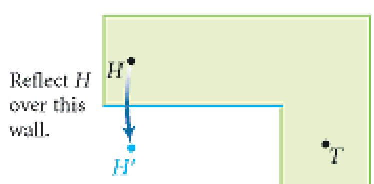


In both cases the path is blocked. It looks like you need to try two bounces. Visualize a path from the tee hitting two walls and into the hole.

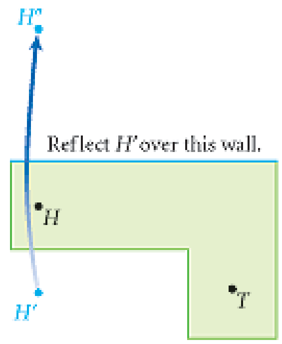


Visualize the path.

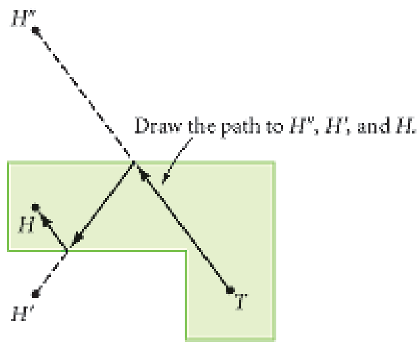
Now you can work backward. Which wall will the ball hit last? Reflect the hole across that wall creating image H' .



Which wall will the ball hit before it approaches the second wall? Reflect the image of the hole H' across that wall creating image H'' (read "H double prime").



Draw the path from the tee to H'', H', and H. Can you visualize other possible paths with two bounces? Three bounces? What do you suppose is the minimal path from T to H?



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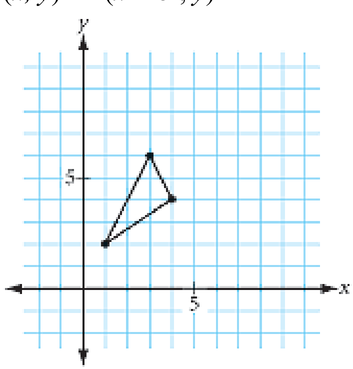
EXERCISES

You will need

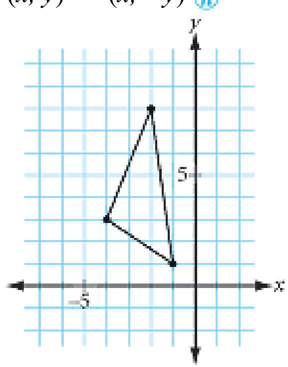
Construction tools for Exercises 19 and 20

In Exercises 1–5, copy the figure and draw the image according to the rule. Identify the type of transformation.

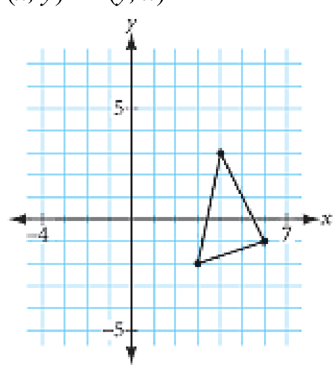
1. $(x, y) \rightarrow (x + 5, y)$



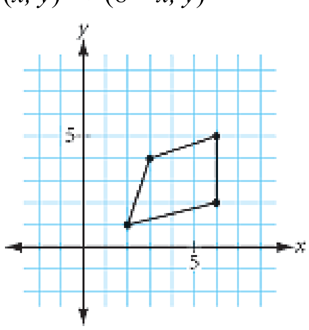
2. $(x, y) \rightarrow (x, -y)$ (h)



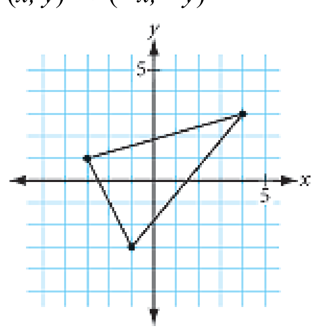
3. $(x, y) \rightarrow (y, x)$



4. $(x, y) \rightarrow (8 - x, y)$



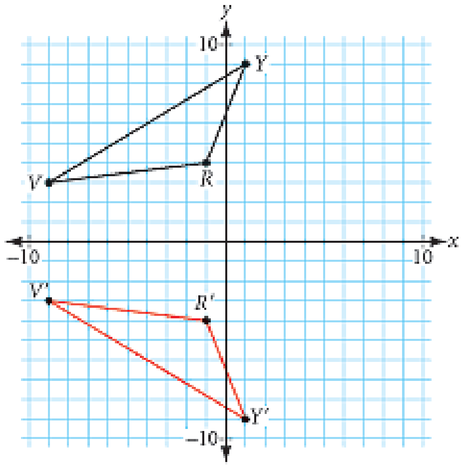
5. $(x, y) \rightarrow (-x, -y)$



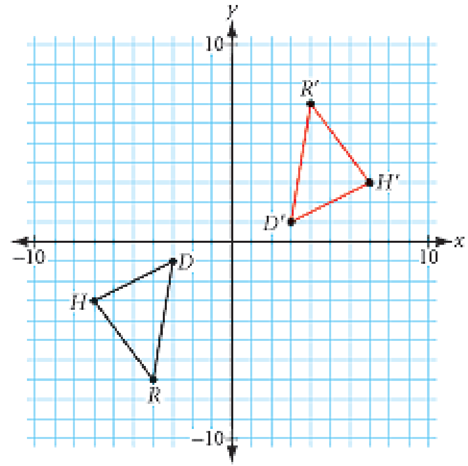
6. Look at the rules in Exercises 1–5 that produced reflections. What do these rules have in common? How about the ones that produce rotations? Translations? For those rules that produce translations, give the translation vector.

In Exercises 7 and 8, complete the ordered pair rule that transforms the black triangle to its image, the red triangle.

7. $(x, y) \rightarrow (\underline{\ ?\ }, \underline{\ ?\ })$ (h)

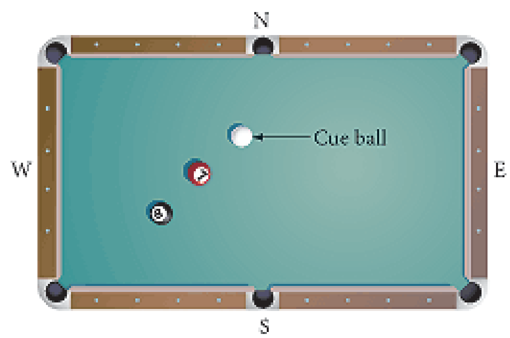


8. $(x, y) \rightarrow (\underline{\ ?\ }, \underline{\ ?\ })$

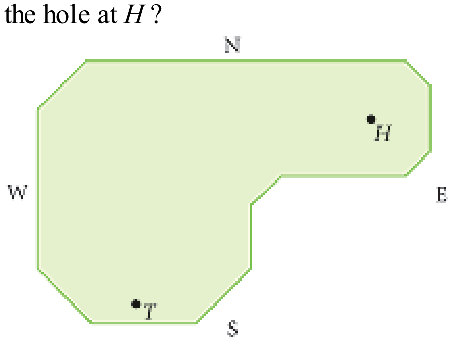


In Exercises 9–11, copy the position of each ball and hole onto patty paper and draw the path of the ball.

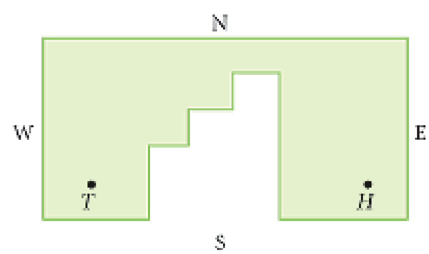
9. What point on the W cushion can a player aim for so that the cue ball bounces and strikes the 8-ball? What point can a player aim for on the S cushion?



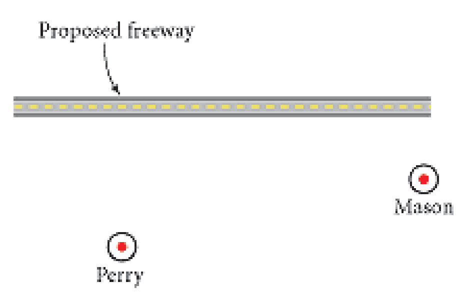
10. Starting from the tee (point T), what point on a wall should a player aim for so that the golf ball bounces off the wall and goes into the hole at H?



11. Starting from the tee (point T), plan a shot so that the golf ball goes into the hole at H. Show all your work.



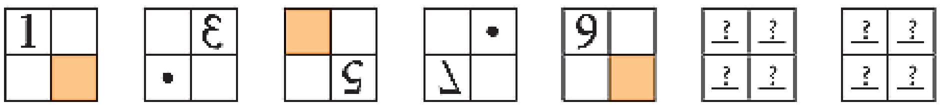
12. A new freeway is being built near the two towns of Perry and Mason. The two towns want to build roads to one junction point on the freeway. (One of the roads will be named Della Street.) Locate the junction point and draw the minimal path from Perry to the freeway to Mason. How do you know this is the shortest path?



Review

In Exercises 13 and 14, sketch the next two figures.

13. 

14. 

15. The word DECODE remains unchanged when it is reflected across its horizontal line of symmetry. Find another such word with at least five letters.

DECODE

16. How many reflectional symmetries does an isosceles triangle have?

17. How many reflectional symmetries does a rhombus have?

18. Write what is actually on the T-shirt shown at right.

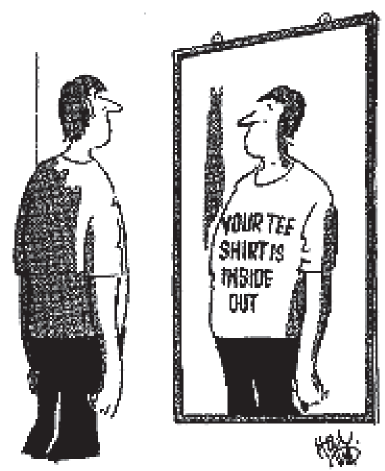
19. ***Construction*** Construct a kite circumscribed about a circle.

20. ***Construction*** Construct a rhombus circumscribed about a circle.

In Exercises 21 and 22, identify each statement as true or false. If true, explain why. If false, give a counterexample.

21. If two angles of a quadrilateral are right angles, then it is a rectangle.

22. If the diagonals of a quadrilateral are congruent, then it is a rectangle.



By Holland. ©1976, Punch Cartoon Library.

IMPROVING YOUR REASONING SKILLS

Chew on This for a While

If the third letter before the second consonant after the third vowel in the alphabet is in the twenty-seventh word of this paragraph, then print the fifteenth word of this paragraph and then print the twenty-second letter of the alphabet after this word. Otherwise, list three uses for chewing gum.

LESSON

7.3

Compositions of Transformations

There are things which nobody would see unless I photographed them.

DIANE ARBUS

In Lesson 7.2, you reflected a point, then reflected it again to find the path of a ball. When you apply one transformation to a figure and then apply another transformation to its image, the resulting transformation is called a **composition** of transformations. Let's look at an example of a composition of two translations.

EXAMPLE

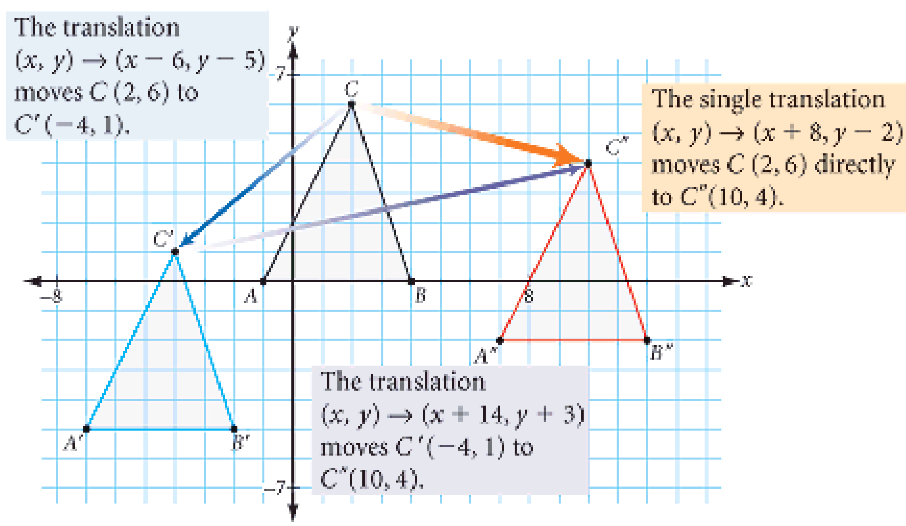
Triangle ABC with vertices $A(-1, 0)$, $B(4, 0)$, and $C(2, 6)$ is first translated by the rule $(x, y) \rightarrow (x - 6, y - 5)$, and then its image, $\triangle A'B'C'$, is translated by the rule $(x, y) \rightarrow (x + 14, y + 3)$.

a. What single translation is equivalent to the composition of these two translations?

b. What single translation brings the second image, $\triangle A''B''C''$, back to the position of the original triangle, $\triangle ABC$?

▶ ***Solution***

Draw $\triangle ABC$ on a set of axes and relocate its vertices using the first rule to get $\triangle A'B'C'$. Then relocate the vertices of $\triangle A'B'C'$ using the second rule to get $\triangle A''B''C''$.



a. Each vertex is moved left 6 then right 14, and down 5 then up 3. So the equivalent single translation would be $(x, y) \rightarrow (x - 6 + 14, y - 5 + 3)$ or $(x, y) \rightarrow (x + 8, y - 2)$. You can also write this as a translation by $\langle 8, -2 \rangle$.

b. Reversing the steps, the translation by $\langle -8, 2\rangle$ brings the second image, $\triangle A'' B'' C''$, back to $\triangle ABC$.

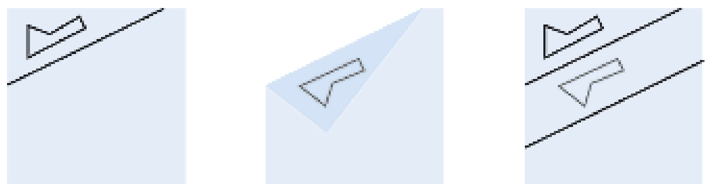
In the investigations you will see what happens when you compose reflections.

Investigation 1
Reflections across Two Parallel Lines

You will need

- patty paper

First, consider the case of parallel lines of reflection.



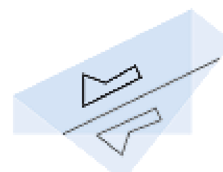
Step 1 Step 2 Step 3

Step 1 On a piece of patty paper, draw a figure and a line of reflection that does not intersect it.

Step 2 Fold to reflect your figure across the line of reflection and trace the image.

Step 3 On your patty paper, draw a second reflection line parallel to the first so that the image is between the two parallel reflection lines.

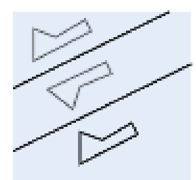
Step 4 Fold to reflect the image across the second line of reflection. Turn the patty paper over and trace the second image.



Step 4

Step 5 How does the second image compare to the original figure? Name the single transformation that transforms the original to the second image.

Step 6 Use a compass or patty paper to measure the distance between a point in the original figure and its second image point. Compare this distance with the distance between the parallel lines. How do they compare?



Step 5

Step 7 Compare your findings with those of others in your group and state your conjecture.

Reflections across Parallel Lines Conjecture C-70

A composition of two reflections across two parallel lines is equivalent to a single __?__. In addition, the distance from any point to its second image under the two reflections is __?__ the distance between the parallel lines.

Is a composition of reflections always equivalent to a single reflection? If you reverse the reflections in a different order, do you still get the original figure back? Can you express a rotation as a set of reflections?

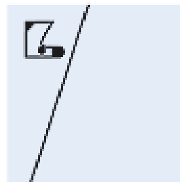
Investigation 2

Reflections across Two Intersecting Lines

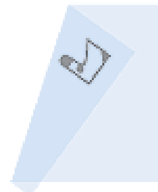
You will need

- patty paper
- a protractor

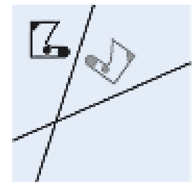
Next, you will explore the case of intersecting lines of reflection.



Step 1



Step 2



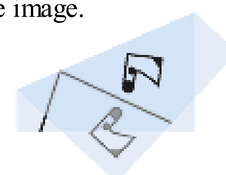
Step 3

Step 1 On a piece of patty paper, draw a figure and a reflection line that does not intersect it.

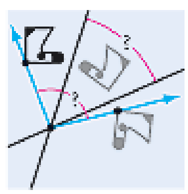
Step 2 Fold to reflect your figure across the line and trace the image.

Step 3 On your patty paper, draw a second reflection line intersecting the first so that the image is in an acute angle between the two intersecting reflection lines.

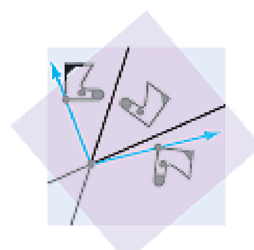
Step 4 Fold to reflect the first image across the second line and trace the second image.



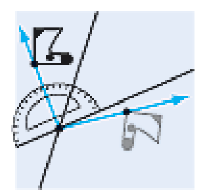
Step 4



Step 5



Step 6



Step 7

Step 5 Draw two rays that start at the point of intersection of the two intersecting lines and that pass through corresponding points on the original figure and its second image.

Step 6 How does the second image compare to the original figure? Name the single transformation from the original to the second image.

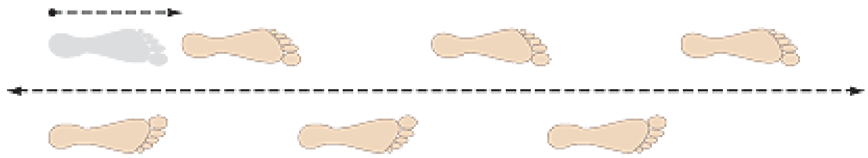
Step 7 With a protractor or patty paper, compare the angle created in Step 5 with the acute angle formed by the intersecting reflection lines. How do the angles compare?

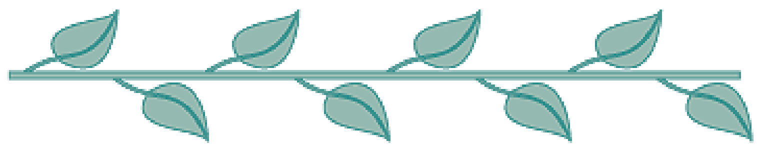
Step 8 Compare findings in your group and state your next conjecture.

Reflections across Intersecting Lines Conjecture   C-71

A composition of two reflections across a pair of intersecting lines is equivalent to a single __?__. The angle of __?__ is __?__ the acute angle between the pair of intersecting reflection lines.

There are many other ways to combine transformations. Combining a translation with a reflection gives a special two-step transformation called a **glide reflection.** A sequence of footsteps is a common example of a glide reflection. You will explore a few other examples of glide reflection in the exercises and later in this chapter.





Glide-reflectional symmetry

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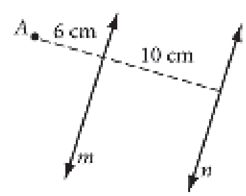
EXERCISES

1. Name the single translation vector that can replace the composition of these three translation vectors: $\langle 2, 3\rangle$, then $\langle -5, 7\rangle$, then $\langle 13, 0\rangle$.

2. Name the single rotation that can replace the composition of these three rotations about the same center of rotation: 45°, then 50°, then 85°. What if the centers of rotation differ? Draw a figure and try it.

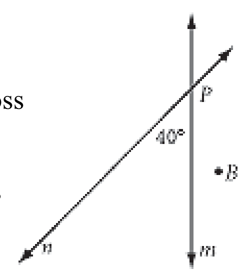
3. Lines m and n are parallel and 10 cm apart.



   a. Point A is 6 cm from line m and 16 cm from line n. Point A is reflected across line m, and then its image, A', is reflected across line n to create a second image, point A''. How far is point A from point A''?

   b. What if A is reflected across n, and then its image is reflected across m? Find the new image and distance from A.

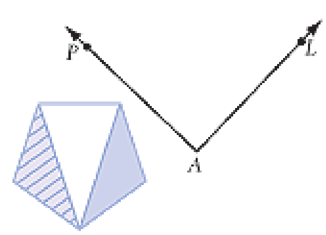
4. Two lines m and n intersect at point P, forming a 40° angle.



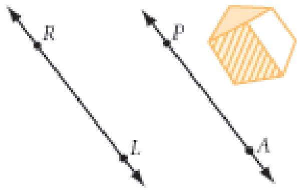
   a. You reflect point B across line m, then reflect the image of B across line n. What angle of rotation about point P rotates the second image of point B back to its original position?

   b. What if you reflect B first across n, and then reflect the image of B across m? Find the angle of rotation that rotates the second image back to the original position.

5. Copy the figure and $\angle PAL$ onto patty paper. Reflect the figure across $\overrightarrow{AP}$. Reflect the image across $\overrightarrow{AL}$. What is the equivalent rotation?

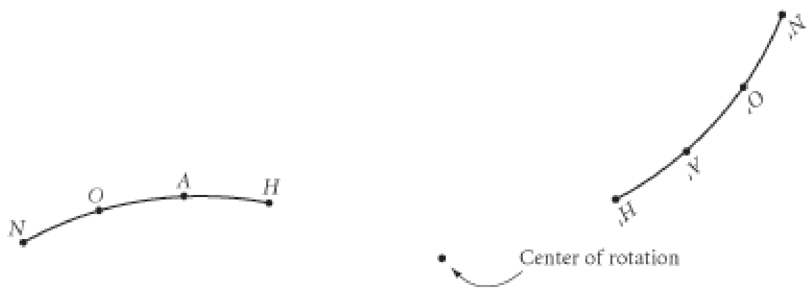


6. Copy the figure and the pair of parallel lines onto patty paper. Reflect the figure across $\overleftrightarrow{PA}$. Reflect the image across $\overleftrightarrow{RL}$. What is the length of the equivalent translation vector?

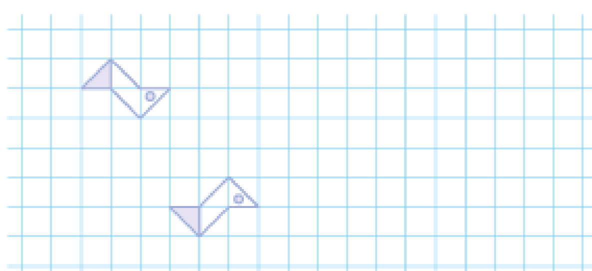


7. Copy the hexagonal figure and its translated image onto patty paper. Find a pair of parallel reflection lines that transform the original onto the image.

8. Copy the original figure and its rotated image onto patty paper. Find a pair of intersecting reflection lines that transform the original onto the image.

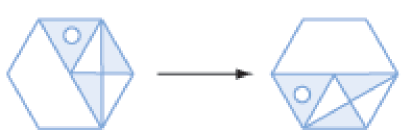


9. Copy the two figures below onto graph paper. Each figure is the glide-reflected image of the other. Continue the pattern with two more glide-reflected figures.



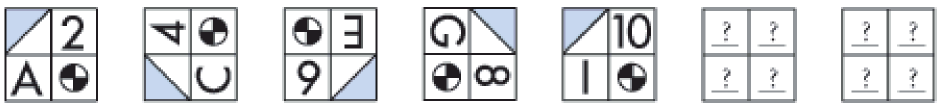
10. Have you noticed that some letters have both horizontal and vertical symmetries? Have you also noticed that all the letters that have both horizontal and vertical symmetries also have point symmetry? Is this a coincidence? Use what you have learned about transformations to explain why.

11. What combination of transformations changed the figure into the image as shown below?

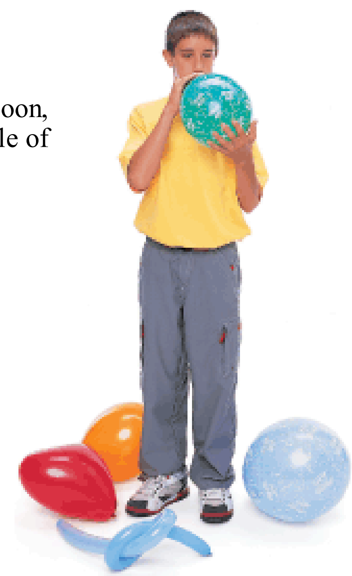


Review

In Exercises 12 and 13, sketch the next two figures.

12. 

13. 



14. If you draw a figure on an uninflated balloon and then blow up the balloon, the figure will undergo a nonrigid transformation. Give another example of a nonrigid transformation.

15. List two objects in your home that have rotational symmetry, but not reflectional symmetry. List two objects in your classroom that have reflectional symmetry, but not rotational symmetry.

16. Is it possible for a triangle to have exactly one line of symmetry? Exactly two? Exactly three? Support your answers with sketches.

17. Draw two points onto a piece of paper and connect them with a curve that is point symmetric. ⓗ

18. Study these two examples of matrix addition, and then use your inductive reasoning skills to fill in the missing entries in a and b.

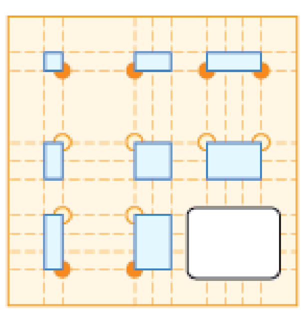
$$\begin{bmatrix} 12 & -7 \\ -14 & 0 \end{bmatrix} + \begin{bmatrix} -8 & -6 \\ 11 & 9 \end{bmatrix} = \begin{bmatrix} 4 & -13 \\ -3 & 9 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 4 \\ 7 & 6 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 5 & -7 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 8 & -3 \\ 5 & 9 & 5 \end{bmatrix}$$

a. $\begin{bmatrix} -5 & 11 \\ -12 & 20 \end{bmatrix} + \begin{bmatrix} \underline{?} & \underline{?} \\ \underline{?} & \underline{?} \end{bmatrix} = \begin{bmatrix} -14 & 11 \\ 0 & 13 \end{bmatrix}$ **b.** $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \begin{bmatrix} a & 2b & 3c \\ -d & d-e & 0 \end{bmatrix} = \begin{bmatrix} \underline{?} & \underline{?} & \underline{?} \\ \underline{?} & \underline{?} & \underline{?} \end{bmatrix}$

IMPROVING YOUR VISUAL THINKING SKILLS

3-by-3 Inductive Reasoning Puzzle I

Sketch the figure missing in the lower-right corner of this 3-by-3 pattern.



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LESSON

7.4

Tessellations with Regular Polygons

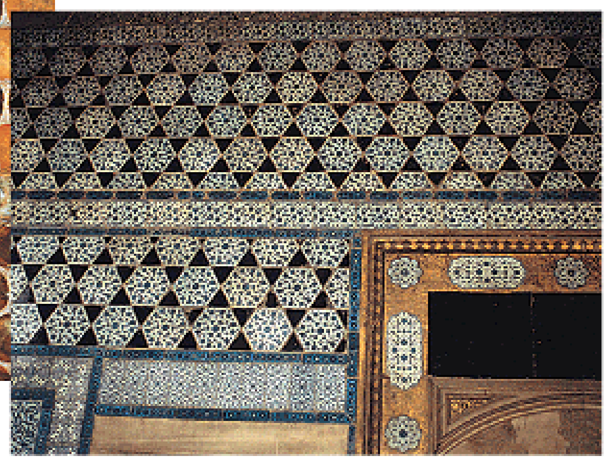
I see a certain order in the universe and math is one way of making it visible.

MAY SARTON

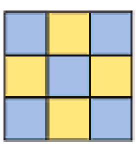
Honeycombs are remarkably geometric structures. The hexagonal cells that bees make are ideal because they fit together perfectly without any gaps. The regular hexagon is one of many shapes that can completely cover a plane without gaps or overlaps. Mathematicians call such an arrangement of shapes a **tessellation** or a **tiling**. A tiling that uses only one shape is called a **monohedral tessellation.**

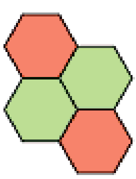
You can find tessellations in every home. Decorative floor tiles have tessellating patterns of squares. Brick walls, fireplaces, and wooden decks often display creative tessellations of rectangles. Where do you see tessellations every day?

The hexagon pattern in the honeycomb of the bee is a tessellation of regular hexagons.



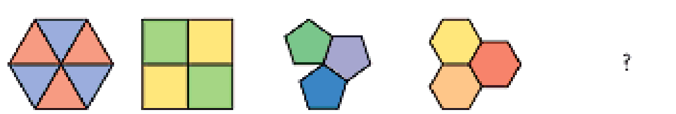
Regular hexagons and equilateral triangles combine in this tiling from the 17th-century Topkapi Palace in Istanbul, Turkey.





You already know that squares and regular hexagons create monohedral tessellations. Because each regular hexagon can be divided into six equilateral triangles, we can logically conclude that equilateral triangles also create monohedral tessellations. Will other regular polygons tessellate? Let's look at this question logically.

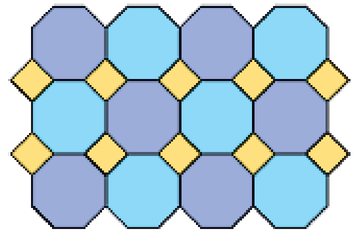
For shapes to fill the plane without gaps or overlaps, their angles, when arranged around a point, must have measures that add up to exactly 360°. If the sum is less than 360°, there will be a gap. If the sum is greater, the shapes will overlap. Six 60° angles from six equilateral triangles add up to 360°, as do four 90° angles from four squares or three 120° angles from three regular hexagons. What about regular pentagons? Each angle in a regular pentagon measures 108°, and 360 is not divisible by 108. So regular pentagons cannot be arranged around a point without overlapping or leaving a gap. What about regular heptagons?



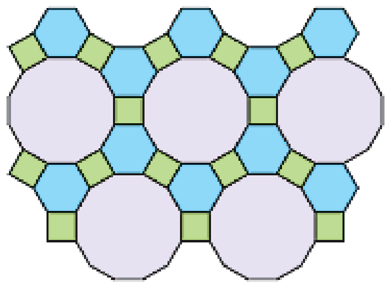
Triangles Squares Pentagons Hexagons Heptagons

In any regular polygon with more than six sides, each angle has a measure greater than 120°, so no more than two angles can fit about a point without overlapping. So the only regular polygons that create monohedral tessellations are equilateral triangles, squares, and regular hexagons. A monohedral tessellation of congruent regular polygons is called a **regular tessellation.**

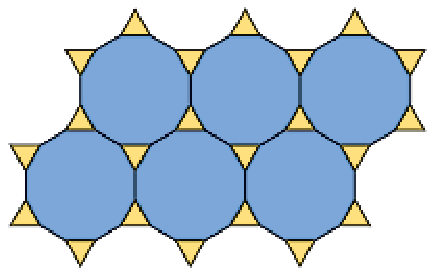
Tessellations can have more than one type of shape. You may have seen the octagon-square combination at right. In this tessellation, two regular octagons and a square meet at each vertex. Notice that you can put your pencil on any vertex and that the point is surrounded by one square and two octagons. So you can call this a 4.8.8 or a 4.8^2 tiling. The sequence of numbers gives the **vertex arrangement,** or **numerical name** for the tiling.



When the same combination of regular polygons (of two or more kinds) meet in the same order at each vertex of a tessellation, it is called a **semiregular tessellation.** Below are two more examples of semiregular tessellations.



The same polygons appear in the same order at each vertex: square, hexagon, dodecagon.



The same polygons appear in the same order at each vertex: triangle, dodecagon, dodecagon.

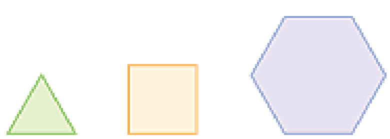
There are eight different semiregular tessellations. Three of them are shown above. In this investigation, you will look for the other five. Fortunately, the remaining five use only combinations of triangles, squares, and hexagons.

Investigation
The Semiregular Tessellations

You will need

- a set of triangles, squares, and hexagons with equal side lengths
- pattern blocks (optional)
- geometry software (optional)

Find or create a set of regular triangles, squares, and hexagons for this investigation. Then work with your group to find the remaining five of the eight semiregular tessellations. Remember, the same combination of regular polygons must meet in the same order at each vertex for the tiling to be semiregular. Also remember to check that the sum of the measures at each vertex is 360°.



Step 1 Investigate which combinations of two kinds of regular polygons you can use to create a semiregular tessellation.

Step 2 Next, investigate which combinations of three kinds of regular polygons you can use to create a semiregular tessellation.

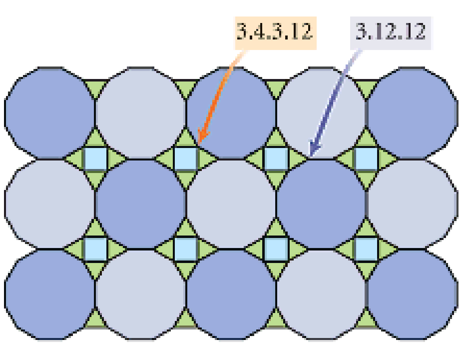
Step 3 Summarize your findings by sketching all eight semiregular tessellations and writing their vertex arrangements (numerical names).

The three regular tessellations and the eight semiregular tessellations you just found are called the **Archimedean tilings.** They are also called 1-uniform tilings because all the vertices in a tiling are identical.

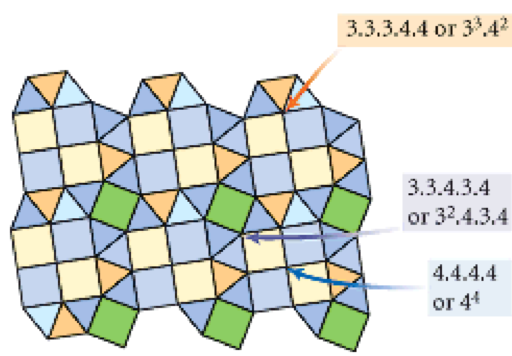
Mathematics CONNECTION

Greek mathematician and inventor Archimedes (ca. 287–212 B.C.E.) studied the relationship between mathematics and art with tilings. He described 11 plane tilings made up of regular polygons, with each vertex being the same type. Plutarch (ca. 46–127 C.E.) wrote of Archimedes' love of geometry, "... he neglected to eat and drink and took no care of his person that he was often carried by force to the baths, and when there he would trace geometrical figures in the ashes of the fire."

Often, different vertices in a tiling do not have the same vertex arrangement. If there are two different types of vertices, the tiling is called 2-uniform. If there are three different types of vertices, the tiling is called 3-uniform. Two examples are shown below.



A 2-uniform tessellation: $3.4.3.12/3.12^2$



A 3-uniform tessellation: $3^3.4^2/3^2.4.3.4/4^4$

There are 20 different 2-uniform tessellations of regular polygons, and 61 different 3-uniform tilings. The number of 4-uniform tessellations of regular polygons is still an unsolved problem.

EXERCISES

You will need

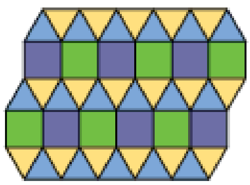
Geometry software for Exercises 11–14

1. Sketch two objects or designs you see every day that are monohedral tessellations.

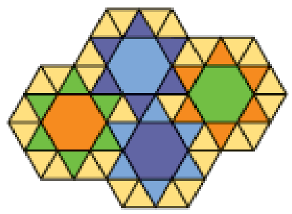
2. List two objects or designs outside your classroom that are semiregular tessellations.

In Exercises 3–5, write the vertex arrangement for each semiregular tessellation in numbers.

3.



4.



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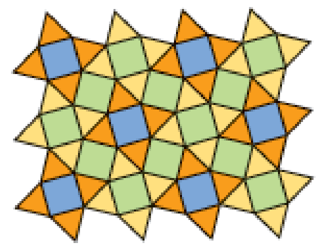
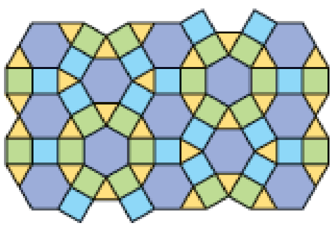
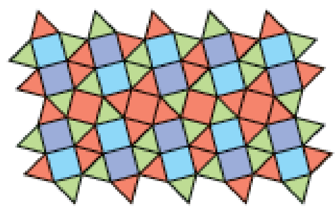


In Exercises 6–8, write the vertex arrangement for each 2-uniform tessellation in numbers.

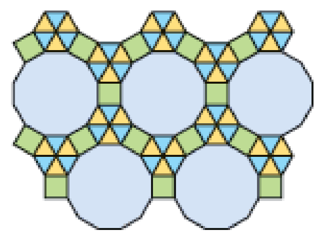
6.



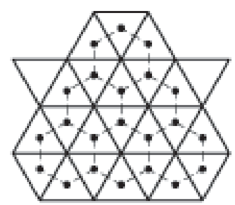
7.



8.



9. When you connect the center of each triangle across the common sides of the tessellating equilateral triangles at right, you get another tessellation. This new tessellation is called the **dual** of the original tessellation. Notice the dual of the equilateral triangle tessellation is the regular hexagon tessellation. Every regular tessellation of regular polygons has a dual.

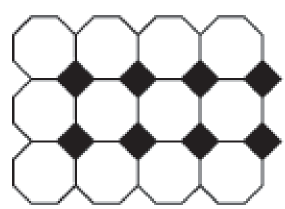


a. Draw a regular square tessellation and make its dual. What is the dual?

b. Draw a hexagon tessellation and make the dual of it. What is the dual?

c. What do you notice about the duals?

10. You can make dual tessellations of semiregular tessellations, but they may not be tessellations of regular polygons. Try it. Sketch the dual of the 4.8.8 tessellation, shown at right. Describe the dual.



Technology In Exercises 11–14, use geometry software, templates of regular polygons, or pattern blocks.

11. Sketch and color the 3.6.3.6 tessellation. Continue it to fill an entire sheet of paper.

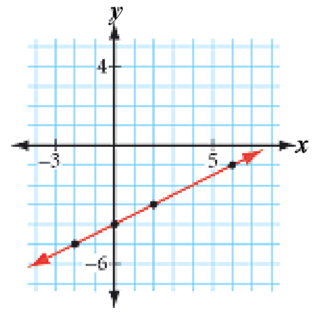
12. Sketch the 4.6.12 tessellation. Color it so it has reflectional symmetry but not rotational symmetry.

13. Show that two regular pentagons and a regular decagon fit about a point, but that 5.5.10 does not create a semiregular tessellation.

14. Create the tessellation 3.12.12/3.4.3.12. Draw your design onto a full sheet of paper. Color your design to highlight its symmetries.

Review

15. Design a logo with rotational symmetry for Happy Time Ice-Cream Company. Or design a logo for your group or for a made-up company.

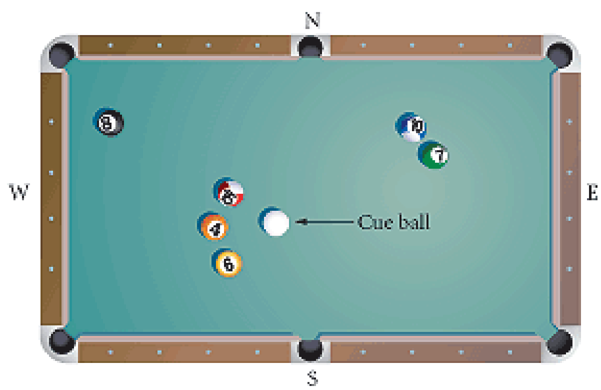


16. Reflect $y = \frac{1}{2}x - 4$ across the x-axis and find the equation of the image line.

17. Words like MOM, WOW, TOOT, and OTTO all have a vertical line of symmetry when you write them in capital letters. Find another word that has a vertical line of symmetry.

The design at left comes from *Inversions,* a book by Scott Kim. Not only does the design spell the word *mirror,* it does so with mirror symmetry!

18. Frisco Fats needs to sink the 8-ball into the NW corner pocket, but he seems trapped. Can he hit the cue ball to a point on the N cushion so that it bounces out, strikes the S cushion, and taps the 8-ball into the corner pocket? Copy the table and construct the path of the ball.



IMPROVING YOUR VISUAL THINKING SKILLS

Painted Faces I

Suppose some unit cubes are assembled into a large cube, then some of the faces of this large cube are painted. After the paint dries, the large cube is disassembled into the unit cubes, and you discover that 32 of these have no paint on any of their faces. How many faces of the large cube were painted?

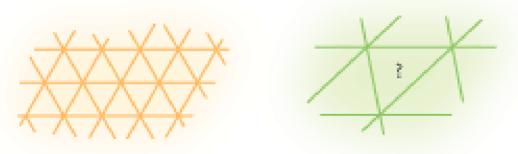
LESSON 7.5

Tessellations with Nonregular Polygons

The most uniquely personal of all that he knows is that which he has discovered for himself.

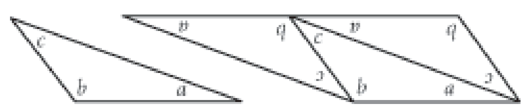
JEROME BRUNER

In Lesson 7.4, you tessellated with regular polygons. You drew both regular and semiregular tessellations with them. What about tessellations of nonregular polygons? For example, will a scalene triangle tessellate? Let's investigate.



Investigation 1
Do All Triangles Tessellate?

Step 1 Make 12 congruent scalene triangles and use them to try to create a tessellation.



Step 2 Look at the angles about each vertex point. What do you notice?

Step 3 What is the sum of the measures of the three angles of a triangle? What is the sum of the measures of the angles that fit around each point? Compare your results with the results of others and state your next conjecture.

Procedure Note

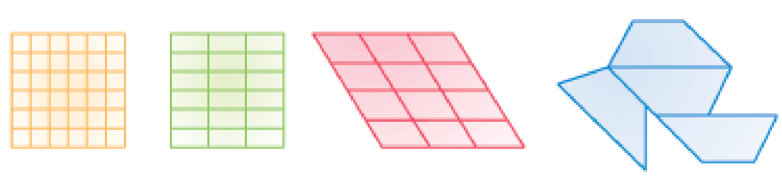
Making Congruent Triangles

1. Stack three pieces of paper and fold them in half.
2. Draw a scalene triangle on the top half-sheet and cut it out, cutting through all six layers to get six congruent scalene triangles.
3. Use one triangle as a template and repeat. You now have 12 congruent triangles.
4. Label the corresponding angles of each triangle *a*, *b*, and *c*, as shown.

Tessellating Triangles Conjecture C-72

? triangle will create a monohedral tessellation.

You have seen that squares and rectangles tile the plane. Can you visualize tiling with parallelograms? Will any quadrilateral tessellate? Let's investigate.



Investigation 2
Do All Quadrilaterals Tessellate?

You want to find out if *any* quadrilateral can tessellate, so you should *not* choose a special quadrilateral for this investigation.

Step 1 Cut out 12 congruent quadrilaterals. Label the corresponding angles in each quadrilateral *a, b, c,* and *d*.

Step 2 Using your 12 congruent quadrilaterals, try to create a tessellation.

Step 3 Notice the angles about each vertex point. How many times does each angle of your quadrilateral fit at each point? What is the sum of the measures of the angles of a quadrilateral? Compare your results with others. State a conjecture.

Tessellating Quadrilaterals Conjecture C-73

__?__ quadrilateral will create a monohedral tessellation.

A regular pentagon does not tessellate, but are there *any* pentagons that tessellate? How many?

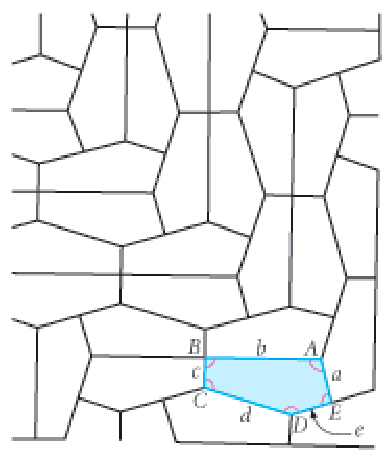
Mathematics CONNECTION

In 1975, when Martin Gardner wrote about pentagonal tessellations in *Scientific American,* experts thought that only eight kinds of pentagons would tessellate. Soon another type was found by Richard James III. After reading about this new discovery, Marjorie Rice began her own investigations.

With no formal training in mathematics beyond high school, Marjorie Rice investigated the tessellating problem and discovered four more types of pentagons that tessellate. Mathematics professor Doris Schattschneider of Moravian College verified Rice's research and brought it to the attention of the mathematics community. Rice had indeed discovered what professional mathematicians had been unable to uncover!

In 1985, Rolf Stein, a German graduate student, discovered a fourteenth type of tessellating pentagon. Are *all* the types of convex pentagons that tessellate now known? The problem remains unsolved.

Shown at right are Marjorie Rice (left) and Dr. Doris Schattschneider.



Type 13, discovered in December 1977

$B = E = 90°,\ 2A + D = 360°$

$2C + D = 360°$

$a = e,\ a + e = d$

One of the pentagonal tessellations discovered by Marjorie Rice. Capital letters represent angle measures in the shaded pentagon. Lowercase letters represent lengths of sides.

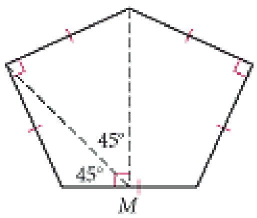
You will experiment with some pentagon tessellations in the exercises.

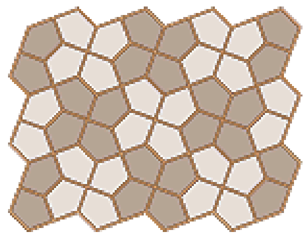
EXERCISES

You will need

Construction tools for Exercise 1

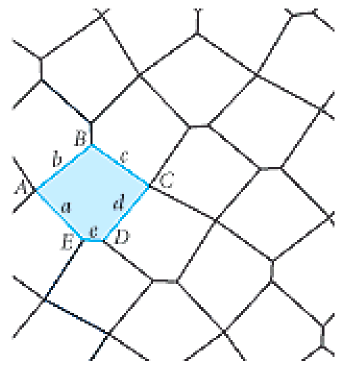
1. ***Construction*** The beautiful Cairo street tiling shown below uses equilateral pentagons. One pentagon is shown below left. Use a ruler and a protractor to draw the equilateral pentagon on poster board or heavy cardboard. (For an added challenge, you can try to *construct* the pentagon, as Egyptian artisans likely would have done.) Cut out the pentagon and tessellate with it. Color your design.





Point M is the midpoint of the base.

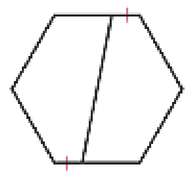
2. At right is Marjorie Rice's first pentagonal tiling discovery. Another way to produce a pentagonal tessellation is to make the dual of the tessellation shown in Lesson 7.4, Exercise 5. Try it. ⓗ



Rice's first discovery, February 1976

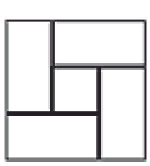
$2E + B = 2D + C = 360°$
$a = b = c = d$

3. A tessellation of regular hexagons can be used to create a pentagonal tessellation by dividing each hexagon as shown. Create this tessellation and color it.

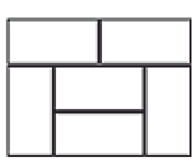


Cultural CONNECTION

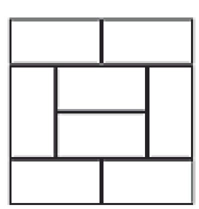
Mats called *tatami* are used as a floor covering in traditional Japanese homes. *Tatami* is made from rush, a flowering plant with soft fibers, and has health benefits, such as removing carbon dioxide and regulating humidity and temperature. When arranging *tatami,* you want the seams to form T-shapes. You avoid arranging four at one vertex forming a cross because it is difficult to get a good fit within a room this way. You also want to avoid fault lines—straight seams passing all the way through a rectangular arrangement—because they make it easier for the *tatami* to slip. Room sizes are often given in *tatami* numbers (for example, a 6-mat room or an 8-mat room).



4.5-mat room

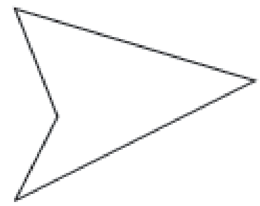


6-mat room



8-mat room

4. Can a concave quadrilateral like the one at right tile the plane? Try it. Create your own concave quadrilateral and try to create a tessellation with it. Decorate your drawing.



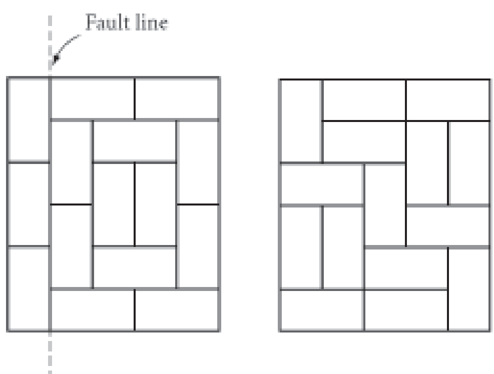
5. Write a paragraph proof explaining why you can use any triangle to create a monohedral tiling.

Review

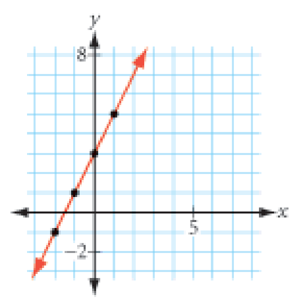
Refer to the Cultural Connection on page 396 for Exercises 6 and 7.

6. Use graph paper to design an arrangement of *tatami* for a 10-mat room. In how many different ways can you arrange the mats so that there are no places where four mats meet at a point (no cross patterns)? Assume that the mats measure 3-by-6 feet and that each room must be at least 9 feet wide. Show all your solutions.

7. There are at least two ways to arrange a 15-mat rectangle with no fault lines. One is shown. Can you find the other?



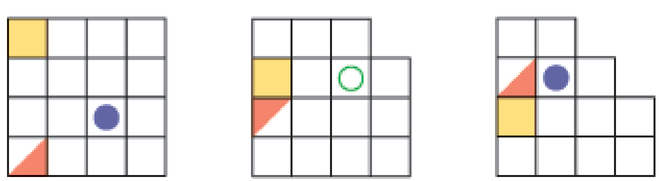
8. Reflect $y = 2x + 3$ across the y-axis and find the equation of the image line.

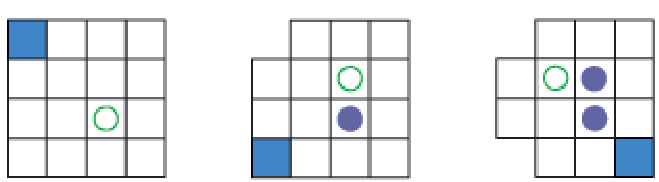


IMPROVING YOUR VISUAL THINKING SKILLS

Picture Patterns II

Draw what comes next in each picture pattern.

1. 

2. 

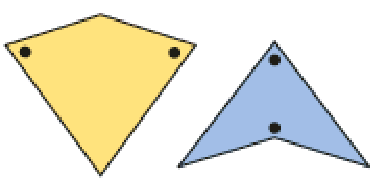
project

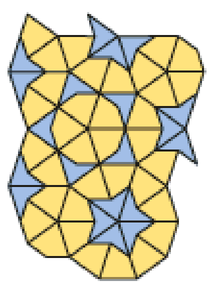
PENROSE TILINGS

When British scientist Sir Roger Penrose of the University of Oxford is not at work on quantum mechanics or relativity theory, he's inventing mathematical games. Penrose came up with a special tiling that uses two shapes, a kite and a dart. The tiles must be placed so that each vertex with a dot always touches only other vertices with dots. By adding this extra requirement, Penrose's tiles make a *nonperiodic tiling*. That is, as you tessellate, the pattern does not repeat by translations.

Penrose tiling at the Center for Mathematics and Computing, Carleton College, Northfield, Minnesota

Try it. Copy the two tiles shown below—the kite and the dart with their dots—onto patty paper. Use the patty-paper tracing to make two cardboard tiles. Create your own unique Penrose tiling and color it. Or, use geometry software to create and color your design. Then answer the questions below.





1. What are the measures of each angle in the kite and the dart?
2. Choose at least three vertices in your design and give the vertex arrangement in terms of kites and darts. How many different combinations of kites and darts can surround a vertex point?
3. Explore how the dots affect the tiling. Why are the dots important?

Your project should include

- Your colored tessellation.
- Your answers to the questions above.

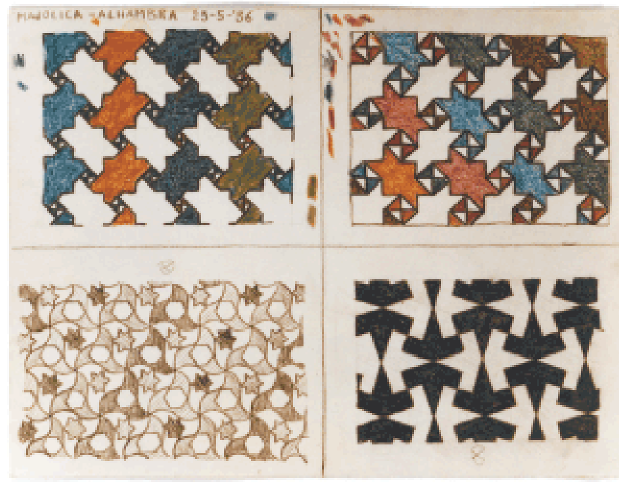
LESSON

7.6

Tessellations Using Only Translations

There are three kinds of people in this world: those who make things happen, those who watch things happen, and those who wonder what happened.
ANONYMOUS

In 1936, M. C. Escher traveled to Spain and became fascinated with the tile patterns of the Alhambra. He spent days sketching the tessellations that Islamic masters had used to decorate the walls and ceilings. Some of his sketches are shown at right. Escher wrote that the tessellations were "the richest source of inspiration" he had ever tapped.



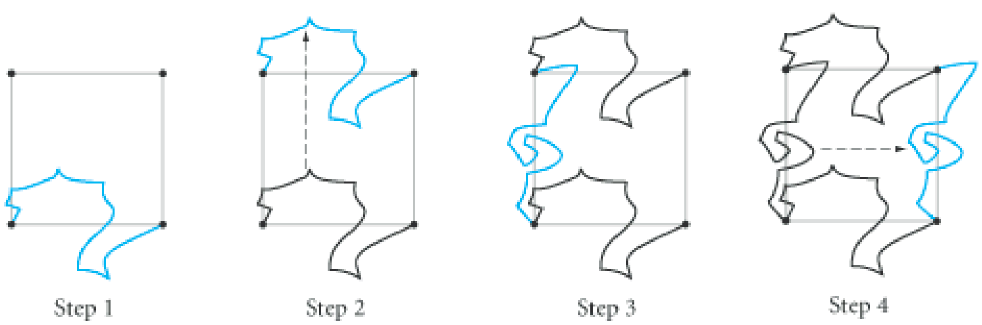
Brickwork, Alhambra, M. C. Escher

Symmetry Drawing E105, M. C. Escher, 1960

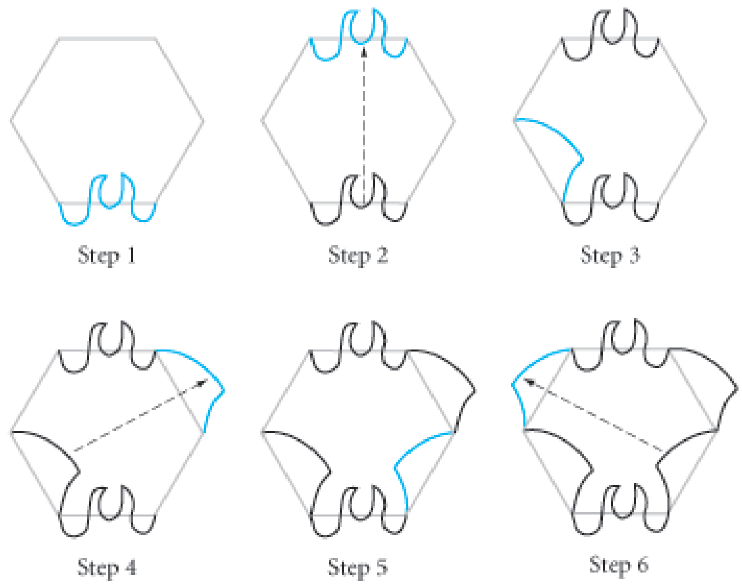
Escher spent many years learning how to use translations, rotations, and glide reflections on grids of equilateral triangles and parallelograms. But he did not limit himself to pure geometric tessellations.

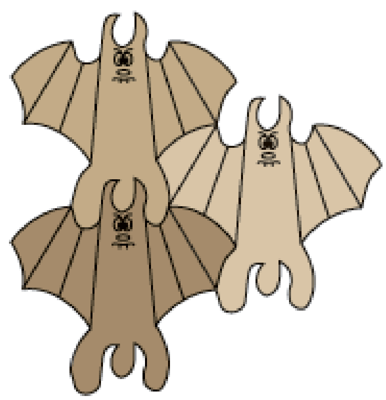
The four steps below show how Escher may have created his Pegasus tessellation, shown at left. Notice how a partial outline of the Pegasus is translated from one side of a square to another to complete a single tile that fits with other tiles like itself.

You can use steps like this to create your own unique tessellation. Start with a tessellation of squares, rectangles, or parallelograms, and try translating curves on opposite sides of the tile. It may take a few tries to get a shape that looks like a person, animal, or plant. Use your imagination!



You can also use the translation technique with regular hexagons. The only difference is that there are three sets of opposite sides on a hexagon. So you'll need to draw three sets of curves and translate them to opposite sides. The six steps below show how student Mark Purcell created his tessellation, *Monster Mix*.





Monster Mix, Mark Purcell

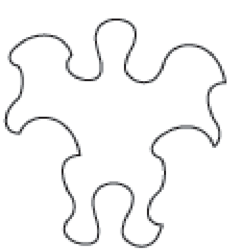
The Escher designs and the student tessellations in this lesson took a great deal of time and practice. When you create your own tessellating designs of recognizable shapes, you'll appreciate the need for this practice! For more about tessellations, and resources to help you learn how to make them, go to www.keymath.com/DG.

Keymath.com Practice Your Skills

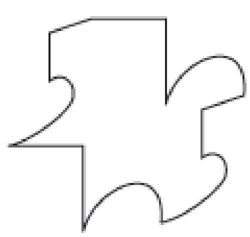
EXERCISES

In Exercises 1–3, copy each tessellating shape and fill it in so that it becomes a recognizable figure.

1.



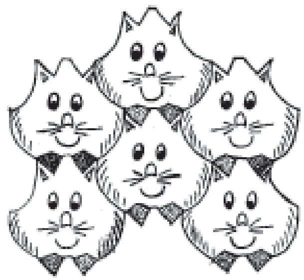
2.



3.

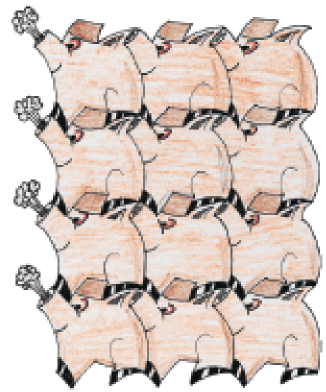
In Exercises 4–6, identify the basic tessellation grid (squares, parallelograms, or regular hexagons) that each geometry student used to create each translation tessellation.

4.



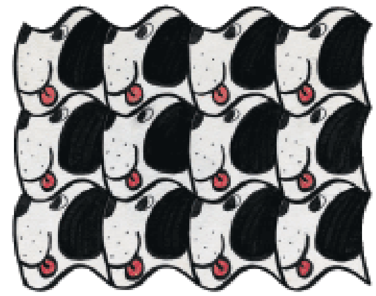
Cat Pack, Renee Chan

5.



Snorty the Pig, Jonathan Benton

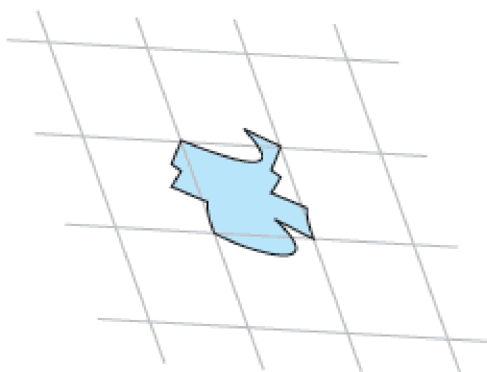
6.



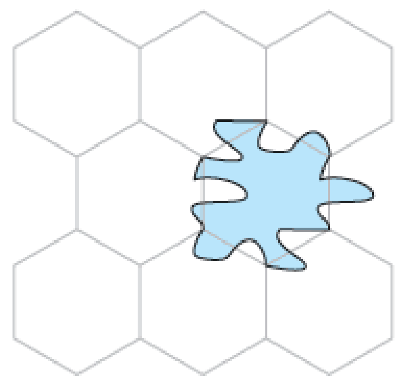
Dog Prints, Gary Murakami

In Exercises 7 and 8, copy the figure and the grid onto patty paper. Create a tessellation on the grid with the figure.

7.



8.



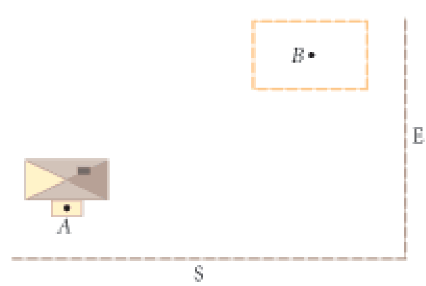
Now it's your turn. In Exercises 9 and 10, create a tessellation of recognizable shapes using the translation method you learned in this lesson. At first, you will probably end up with shapes that look like amoebas or spilled milk, but with practice and imagination, you will get recognizable images. Decorate and title your designs.

9. Use squares as the basic structure. ⓗ

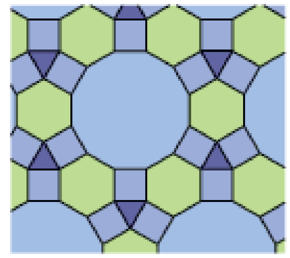
10. Use regular hexagons as the basic structure.

Review

11. The route of a rancher takes him from the house at point A to the south fence, then over to the east fence, then to the corral at point B. Copy the figure at right onto patty paper and locate the points on the south and east fences that minimize the rancher's route.



12. Reflect $y = \frac{2}{3}x + 3$ across the y-axis. Write an equation for the image. How does it compare with the original equation?



13. Give the vertex arrangement for the tessellation at right.

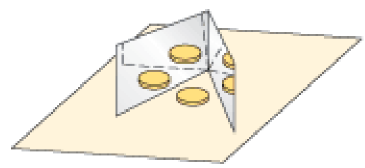
14. A helicopter has four blades. Each blade measures about 28 feet from the center of rotation to the tip. What is the speed in feet per second at the tips of the blades when they are moving at 440 rpm?

15. *Developing Proof* Identify each of the following statements as true or false. If true, explain why. If false, give a counterexample explaining why it is false.
    a. If the two diagonals of a quadrilateral are congruent, but only one is the perpendicular bisector of the other, then the quadrilateral is a kite.
    b. If the quadrilateral has exactly one line of reflectional symmetry, then the quadrilateral is a kite.
    c. If the diagonals of a quadrilateral are congruent and bisect each other, then it is a square.
    d. If a trapezoid is cyclic, then it is isosceles.

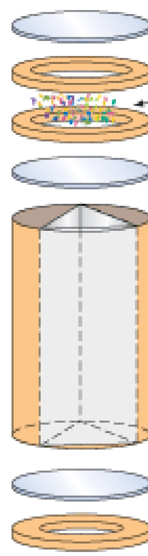
project

KALEIDOSCOPES

You have probably looked through kaleidoscopes and enjoyed their beautiful designs, but do you know how they work? For a simple kaleidoscope, hinge two mirrors with tape, place a small object or photo between the mirrors, and adjust them until you see four objects (the original and three images). What is the angle between the mirrors? At what angle should you hold the mirrors to see six objects? Eight objects?



The British physicist Sir David Brewster invented the tube kaleidoscope in 1816. Some tube kaleidoscopes have colored glass or plastic pieces that tumble around in their end chambers. Some have colored liquid. Others have only a lens in the chamber—the design you see depends on where you aim it.



Design and build your own kaleidoscope using a plastic or cardboard cylinder and glass or plastic as reflecting surfaces. Try various items in the end chamber.

Your project should include

- Your kaleidoscope (pass it around!).
- A report with diagrams that show the geometry properties you used, a list of the materials and tools you used, and a description of problems you had and how you solved them.

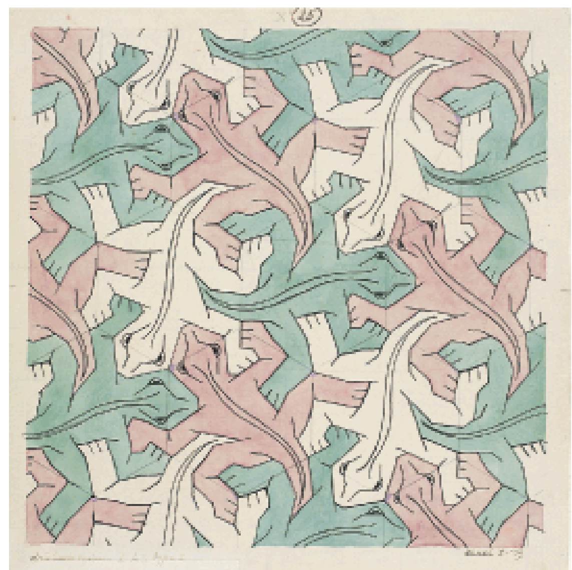
LESSON 7.7

Tessellations That Use Rotations

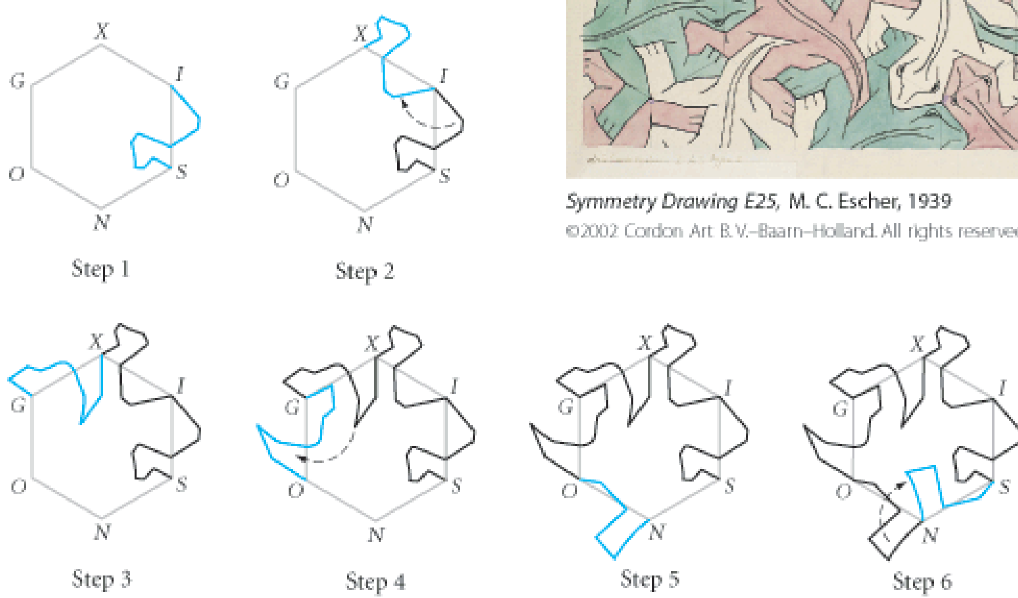
Einstein was once asked, "What is your phone number?" He answered, "I don't know, but I know where to find it if I need it."

ALBERT EINSTEIN

In Lesson 7.6, you created recognizable shapes by translating curves from opposite sides of a regular hexagon or square. In tessellations using only translations, all the figures face in the same direction. In this lesson you will use rotations of curves on a grid of parallelograms, equilateral triangles, or regular hexagons. The resulting tiles will fit together when you rotate them, and the designs will have rotational symmetry about points in the tiling. For example, in this Escher print, each reptile is made by rotating three different curves about three alternating vertices of a regular hexagon.



Symmetry Drawing E25, M. C. Escher, 1939



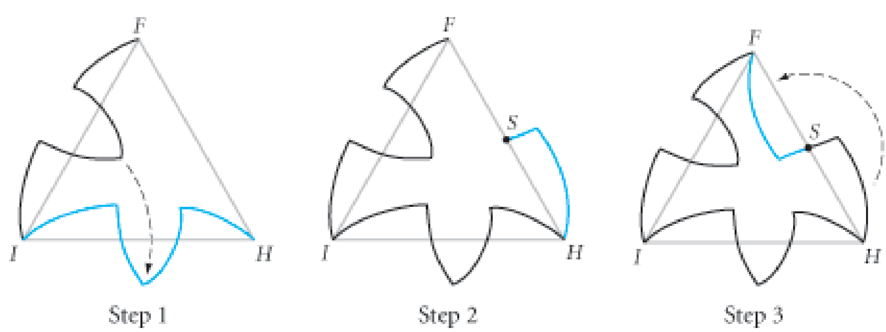
Step 1	Connect points S and I with a curve.
Step 2	Rotate curve SI about point I so that point S rotates to coincide with point X.
Step 3	Connect points G and X with a curve.
Step 4	Rotate curve GX about point G so that point X rotates to coincide with point O.
Step 5	Create curve NO.
Step 6	Rotate curve NO about point N so that point O rotates to coincide with point S.

Escher worked long and hard to adjust each curve until he got what he recognized as a reptile. When you are working on your own design, keep in mind that you may have to redraw your curves a few times until something you recognize appears.

Escher used his reptile drawing in this famous lithograph. Look closely at the reptiles in the drawing. Escher loved to play with our perceptions of reality!

Reptiles, M. C. Escher, 1943

Another method used by Escher utilizes rotations on an equilateral triangle grid. Two sides of each equilateral triangle have the same curve, rotated about their common point. The third side is a curve with point symmetry. The following steps demonstrate how you might create a tessellating flying fish like that created by Escher.

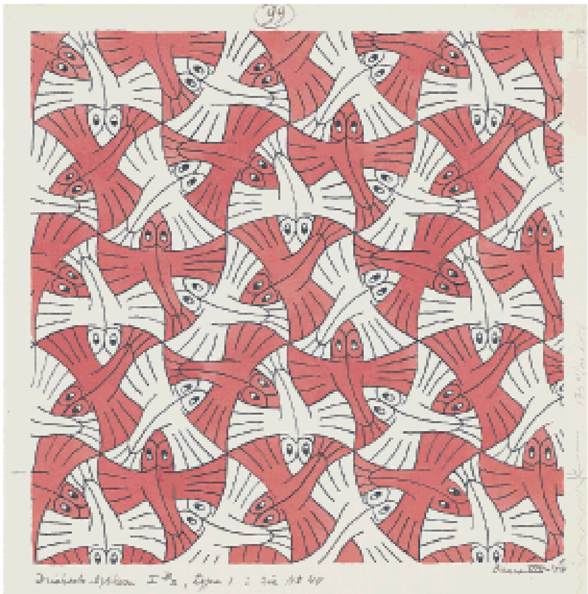


Step 1 Connect points F and I with a curve. Then rotate the curve 60° clockwise about point I so that it becomes curve IH.

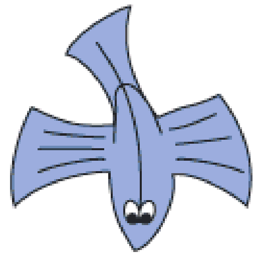
Step 2 Find the midpoint S of $\overline{FH}$ and draw curve SH.

Step 3 Rotate curve SH 180° about S to produce curve FS. Together curve FS and curve SH become the point-symmetric curve FH.

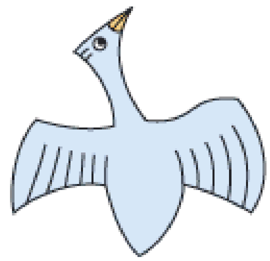
With a little added detail, the design becomes a flying fish.



Symmetry Drawing E99,
M. C. Escher, 1954



Or, with just a slight variation in the curves, the resulting shape will appear more like a bird than a flying fish.



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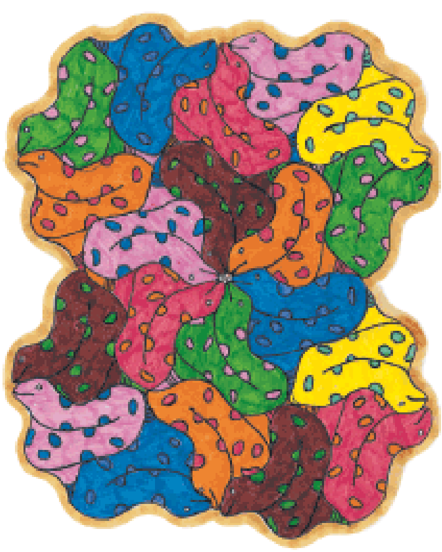
EXERCISES

You will need

Geometry software for Exercise 13

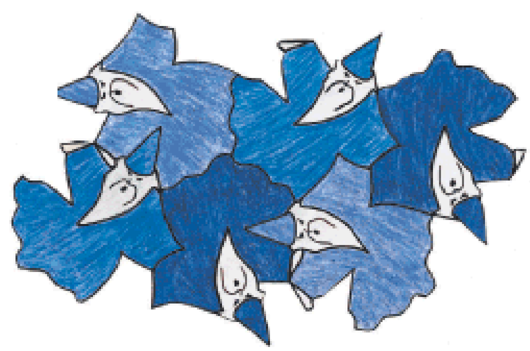
In Exercises 1 and 2, identify the basic grid (equilateral triangles or regular hexagons) that each geometry student used to create the tessellation.

1.



Snakes, Jack Chow

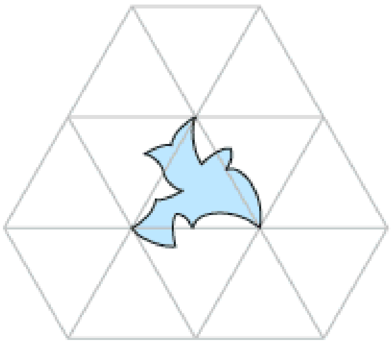
2.



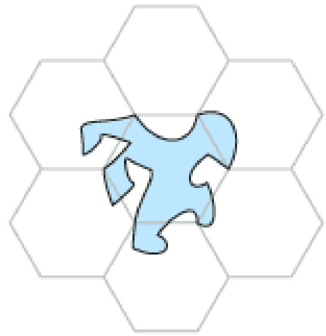
Merlin, Aimee Plourdes

In Exercises 3 and 4, copy the figure and the grid onto patty paper. Show how you can use other pieces of patty paper to tessellate the figure on the grid.

3.



4.



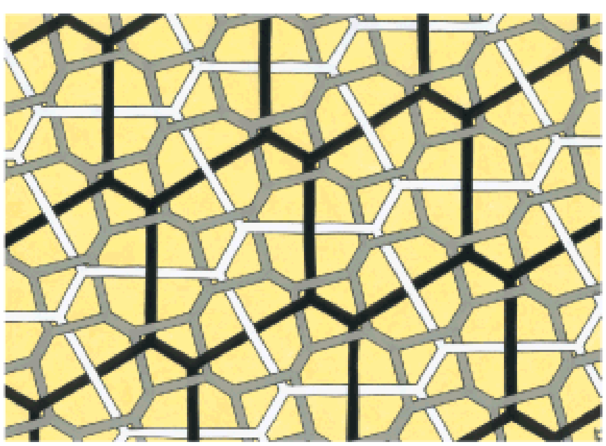
In Exercises 5 and 6, create tessellation designs by using rotations. You will need patty paper, tracing paper, or clear plastic, and grid paper or isometric dot paper.

5. Create a tessellating design of recognizable shapes by using a grid of regular hexagons. Decorate and color your art.

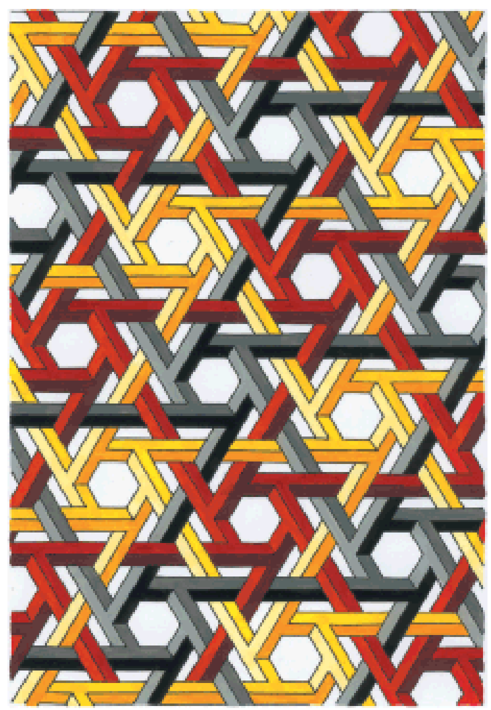
6. Create a tessellating design of recognizable shapes by using a grid of equilateral or isosceles triangles. Decorate and color your art.

Review

7. Study these knot designs by Rinus Roelofs. Now try creating one of your own. Select a tessellation. Make a copy and thicken the lines. Make two copies of this thick-lined tessellation. Lay one of them on top of the other and shift it slightly. Trace the one underneath onto the top copy. Erase where they overlap. Then create a knot design using what you learned in Lesson 0.5.



Dutch artist Rinus Roelofs (b 1954) experiments with the lines between the shapes rather than looking at the plane-filling figures. In these paintings, he has made the lines thicker and created intricate knot designs.



(Above) *Impossible Structures–III, structure 24*
(At left) *Interwoven Patterns–V, structure 17*

For Exercises 8–11, identify the statement as true or false. For each false statement, explain why it is false or sketch a counterexample.

8. If the diagonals of a quadrilateral are congruent, the quadrilateral is a parallelogram.

9. If the diagonals of a quadrilateral are congruent and bisect each other, the quadrilateral is a rectangle.

10. If the diagonals of a quadrilateral are perpendicular and bisect each other, the quadrilateral is a rhombus.

11. If the diagonals of a quadrilateral are congruent and perpendicular, the quadrilateral is a square.

12. Earth's radius is about 4000 miles. Imagine that you travel from the equator to the South Pole by a direct route along the surface. Draw a sketch of your path. How far will you travel? How long will the trip take if you travel at an average speed of 50 miles per hour?

13. ***Technology*** Use geometry software to construct a line and two points A and B not on the line. Reflect A and B over the line and connect the four points to form a trapezoid.

a. Explain why it is a trapezoid.

b. Is it an isosceles trapezoid? Explain.

c. Choose a random point C inside the trapezoid and connect it to the four vertices with segments. Calculate the sum of the distances from C to the four vertices. Drag point C around. Where is the sum of the distances the greatest? The least?

14. Study these two examples of matrix multiplication, and then use your inductive reasoning skills to fill in the missing entries in a and b.

$$\begin{bmatrix} 5 & 3 \\ 8 & 7 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 14 & -7 \end{bmatrix} \qquad \begin{bmatrix} 4 & 20 & 2 \\ 7 & 30 & 1 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & -5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 46 & -100 \\ 73 & -150 \end{bmatrix}$$

a. $$\begin{bmatrix} 3 & 5 & -6 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 8 & 7 \\ 6 & 0 \\ -9 & 2 \end{bmatrix} = \begin{bmatrix} \underline{?} & \underline{?} \\ \underline{?} & \underline{?} \end{bmatrix}$$

b. $$\begin{bmatrix} 8 & 3 \\ 12 & -5 \end{bmatrix}\begin{bmatrix} \underline{2} & \underline{?} \\ \underline{?} & \underline{?} \end{bmatrix} = \begin{bmatrix} 13 & 30 \\ \underline{?} & -50 \end{bmatrix}$$

IMPROVING YOUR REASONING SKILLS

Logical Liars

Five students have just completed a logic contest. To confuse the school's reporter, Lois Lang, each student agreed to make one true and one false statement to her when she interviewed them. Lois was clever enough to figure out the winner. Are you? Here are the students' statements.

Frances: Kai was second. I was fourth.
Leyton: I was third. Charles was last.
Denise: Kai won. I was second.
Kai: Leyton had the best score. I came in last.
Charles: I came in second. Kai was third.

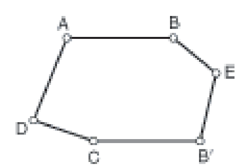
Exploration

Tessellating with the Conway Criterion

You've discovered that you can tessellate with any triangle, any quadrilateral, some nonregular pentagons, and regular hexagons. The Conway criterion, named for English mathematician John Horton Conway (b 1937), describes rules for tiles that will always tessellate. The sides of a tile satisfying the Conway criterion need not be straight, but they must have certain characteristics that you'll discover in this activity. You'll use Sketchpad to tessellate with a Conway-criterion hexagon, and then you'll experiment with special cases.

Activity

Conway Hexagons



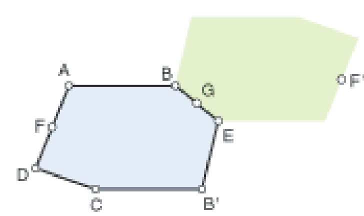
Step 1 Construct segment AB and point C not on $\overline{AB}$.

Step 2 Select point A and point C in order, and choose **Mark Vector** from the Transform menu.

Step 3 Select $\overline{AB}$ and then point B, and choose **Translate** from the Transform menu. You now have a pair of parallel, congruent sides.

Step 4 Construct points D and E not on $\overline{AB}$ or $\overline{CB'}$, and then construct $\overline{AD}$, $\overline{DC}$, $\overline{BE}$, and $\overline{EB'}$. Do you think this hexagon will tessellate?

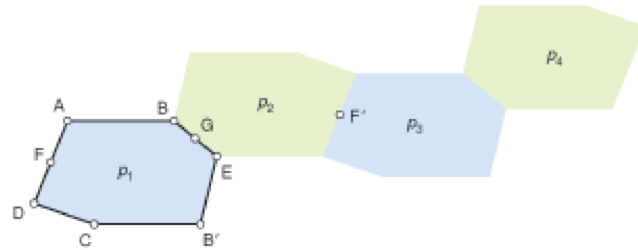
Step 5 Construct midpoint F of $\overline{AD}$ and midpoint G of $\overline{BE}$.



Step 6 Select the vertices of the hexagon in consecutive order, and choose **Hexagon Interior** from the Construct menu.

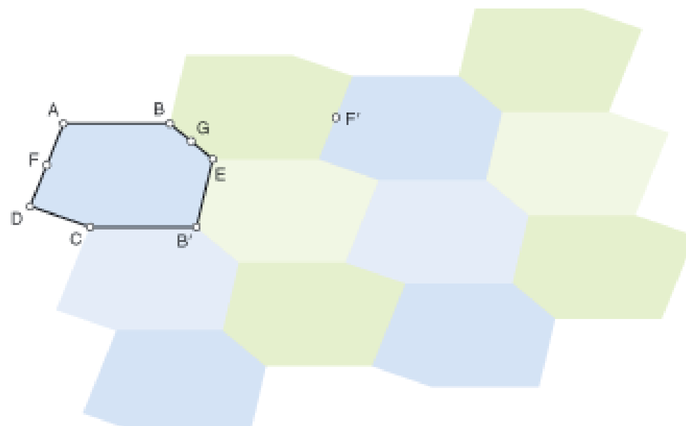
Step 7 Double-click point G to mark it as a center. Select the polygon interior and point F, and choose **Rotate** from the Transform menu. Rotate the objects 180°. Give the image polygon a different color.

Step 8 Mark vector FF'. Select the two polygon interiors and translate them by the marked vector. You should now have a row of polygons. Drag any of the points. How do the polygons fit together?



Step 9 Call the polygons from left to right p_1, p_2, p_3, and p_4. Polygon p_3 is the translation of polygon p_1 by the vector FF'. How is polygon p_3 related to polygon p_2? How is polygon p_2 related to polygon p_1? What transformation is equivalent to the composition of two 180° rotations about two different centers?

Step 10 Mark vector AC. Select the row of polygon interiors and translate by this marked vector. Translate again one or two more times. Color the polygons so that you can tell them apart.



Step 11 Not all hexagons tessellate, but a conjecture was made that those with one pair of opposite sides congruent and parallel do tessellate. Drag any of the vertices. What do you think of this conjecture? The conjecture is called the Conway criterion for hexagons.

Step 12 Drag point B on top of point A. Now what kind of tessellating shape do you have? What is it about the angles of this shape that guarantee it will tessellate? Is the Conway criterion needed to guarantee that these figures tessellate?

Step 13 With point B still on top of point A, drag point E on top of points B and A. Now what shape do you have? What is it about the angles of this shape that guarantee it will tessellate?

Step 14 Undo so that you have a hexagon again. Make the hexagon into a pentagon that will tessellate. Explain what you did.

Step 15 Undo so that you have a hexagon again. In any tessellation the angle measures at each vertex must sum to 360°. Does the Conway criterion guarantee that the sum of the angle measures equals 360°? Explain. (*Hint:* Construct $\overline{AC}$ and $\overline{BB'}$, and construct an auxiliary line parallel to $\overline{BB'}$ through point E.)

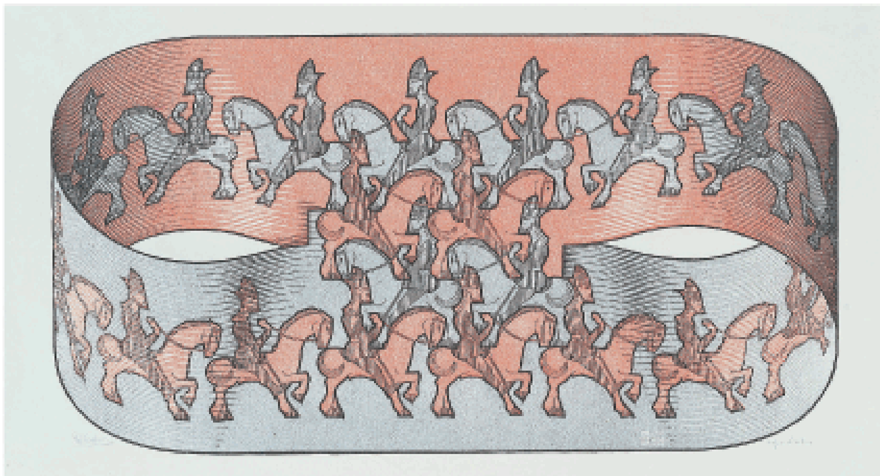
Student Web Links @ Keymath.com

LESSON

7.8 Tessellations That Use Glide Reflections

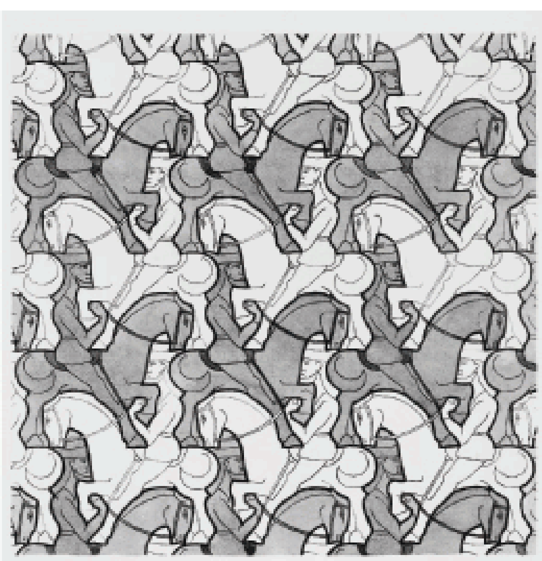
The young do not know enough to be prudent, and therefore they attempt the impossible, and achieve it, generation after generation.
PEARL S. BUCK

In this lesson you will use glide reflections to create tessellations. In Lesson 7.6, you saw Escher's translation tessellation of the winged horse Pegasus. All the horses are facing in the same direction. In the drawings below and below left, Escher used glide reflections on a grid of glide-reflected kites to get his horsemen facing in opposite directions.

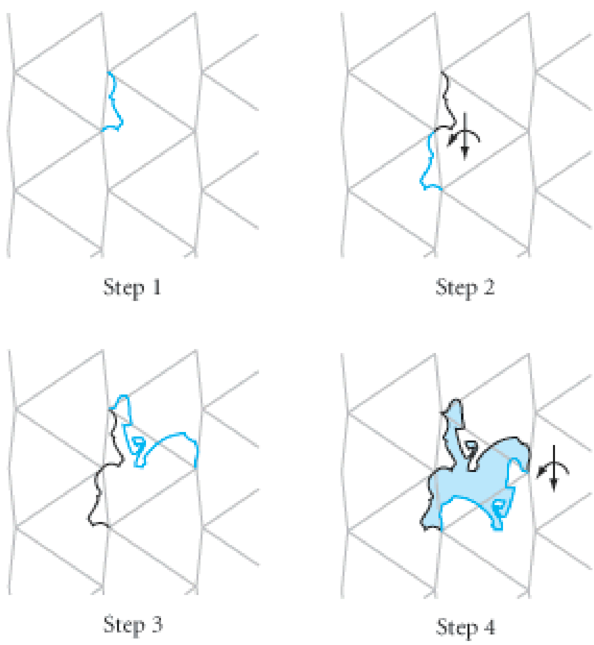


Horseman, M. C. Escher, 1946

The steps below show how you can make a tessellating design similar to Escher's *Horseman*. (The symbol indicates a glide reflection.)

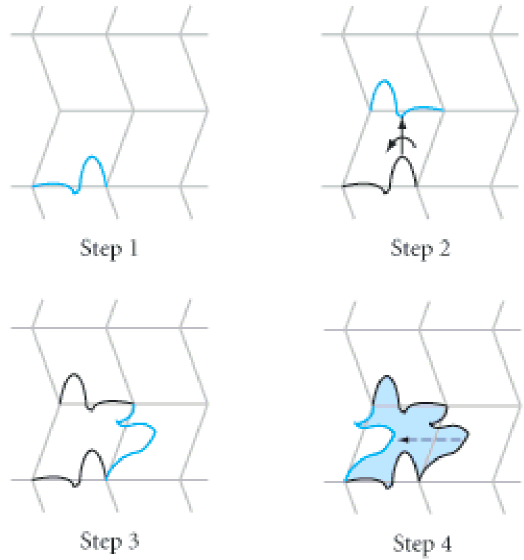


Horseman Sketch, M. C. Escher



In the tessellation of birds below left, you can see that Escher used a grid of squares. You can use the same procedure on a grid of any type of glide-reflected parallelograms. The steps below show how you might create a tessellation of birds or fishes on a parallelogram grid.

Symmetry Drawing E108, M. C. Escher, 1967



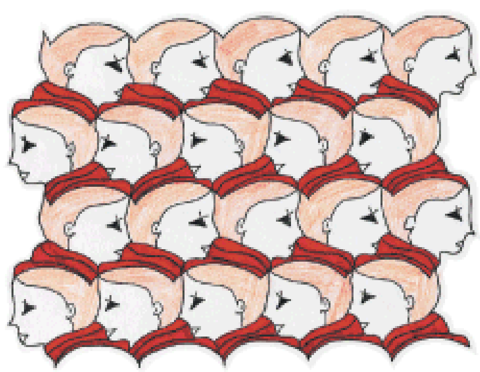
EXERCISES

You will need

Construction tools for Exercise 8

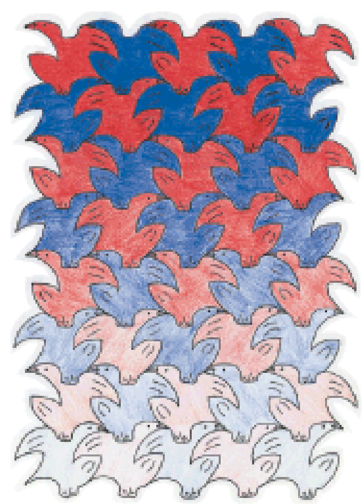
In Exercises 1 and 2, identify the basic tessellation grid (kites or parallelograms) that the geometry student used to create the tessellation.

1.



A Boy with a Red Scarf, Elina Uzin

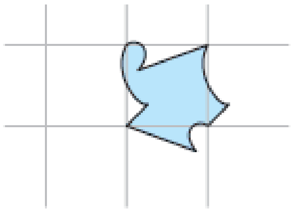
2.



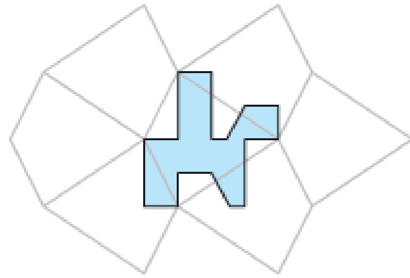
Glide Reflection, Alice Chan

In Exercises 3 and 4, copy the figure and the grid onto patty paper. Show how you can use other patty paper to tessellate the figure on the grid.

3.



4.



5. Create a glide-reflection tiling design of recognizable shapes by using a grid of kites. Decorate and color your art.

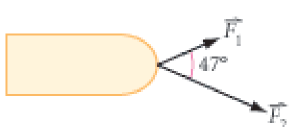
6. Create a glide-reflection tiling design of recognizable shapes by using a grid of parallelograms. Decorate and color your art.

Review

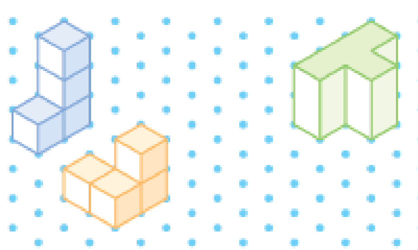
7. Find the coordinates of the circumcenter and orthocenter of $\triangle FAN$ with $F(6, 0)$, $A(7, 7)$, and $N(3, 9)$.

8. ***Construction*** Construct a circle and a chord of the circle. With compass and straightedge, construct a second chord parallel and congruent to the first chord.

9. Remy's friends are pulling him on a sled. One of his friends is stronger and exerts more force. The vectors in this diagram represent the forces his two friends exert on him. Copy the vectors, complete the vector parallelogram, and draw the resultant vector force on his sled.



10. The green prism below right was built from the two solids below left. Copy the figure on the right onto isometric dot paper and shade in one of the two pieces to show how the complete figure was created.



IMPROVING YOUR ALGEBRA SKILLS

Fantasy Functions

If $a \circ b = a^b$ then $3 \circ 2 = 3^2 = 9$,

and if $a \Delta b = a^2 + b^2$ then $5 \Delta 2 = 5^2 + 2^2 = 29$.

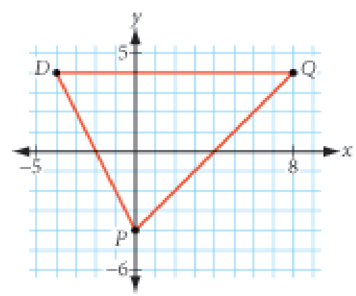
If $8 \Delta x = 17 \circ 2$, find x.

Finding Points of Concurrency

Suppose you know the coordinates of the vertices of a triangle. In the previous chapter, you saw that you can find the coordinates of the circumcenter by writing equations for the perpendicular bisectors of two of the sides and solving the system. Similarly, you can find the coordinates of the orthocenter by finding equations for two lines containing altitudes ofthe triangle and solving the system.

EXAMPLE A

Find the coordinates of the orthocenter of $\triangle PDQ$ with $P(0, -4)$, $D(-4, 4)$, and $Q(8, 4)$.



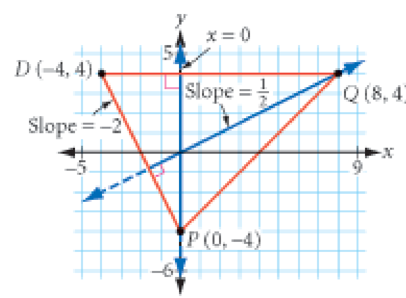
▸ Solution

The orthocenter is the intersection of two altitudes of the triangle. An altitude passes through a vertex and is perpendicular to the opposite side. Because $\overline{DQ}$ is horizontal, the altitude from P to $\overline{DQ}$ must be a vertical line. The line that is vertical and passes through P is the line $x = 0$.

Next, find the equation of the line containing one of the other altitudes. You can find the equation of the line containing the altitude from Q to $\overline{DP}$ by finding the line that is perpendicular to $\overline{DP}$ and that passes through Q.

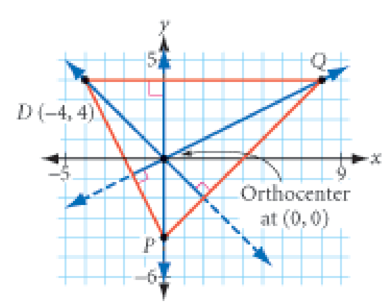
The slope of $\overline{DP}$ is $\frac{-4-4}{0-(-4)}$, or -2. The altitude is perpendicular to $\overline{DP}$, so the slope of the altitude is $\frac{1}{2}$, the opposite reciprocal of -2. Using the definition of slope and the coordinates of Q, the equation of the dashed line is $\frac{y-4}{x-8} = \frac{1}{2}$. Solving for y gives $y = \frac{1}{2}x$ as the equation of the line containing the altitude.

For a review of this technique of using the definition of slope to write an equation, see Example C on pages 288–289.



To find the point where the altitudes intersect, solve the system of equations. You already know $x = 0$. If you substitute 0 for x into $y = \frac{1}{2}x$, you get $y = 0$. So, the orthocenter is $(0, 0)$.

You can verify this result by writing the equation of the line containing the third altitude and making sure $(0, 0)$ satisfies it.



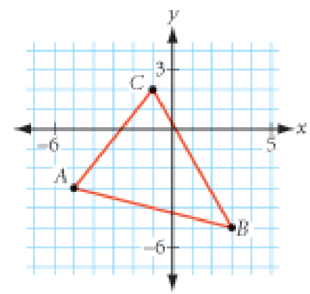
You can also find the coordinates of the centroid of a triangle by solving a system of two lines containing medians. However, as you will see in the next example, there is a more efficient method.

EXAMPLE B

Consider $\triangle ABC$ with $A(-5, -3)$, $B(3, -5)$, and $C(-1, 2)$.

a. Find the coordinates of the centroid of $\triangle ABC$ by writing equations for two lines containing medians and finding their point of intersection.

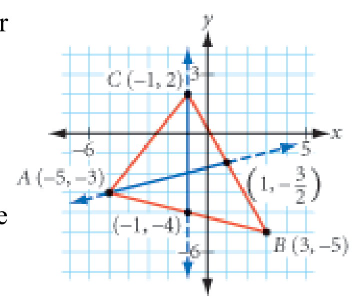
b. Find the mean of the x-coordinates and the mean of the y-coordinates of the triangle's vertices. What do you notice?



Solution

The centroid is the intersection of two medians of the triangle. A median joins a vertex with the midpoint of the opposite side.

a. The midpoint of $\overline{AB}$ is $\left(\frac{-5+3}{2}, \frac{-3+(-5)}{2}\right)$ or $(-1, -4)$. The coordinates of C are $(-1, 2)$, so the line that goes through both of these points is the vertical line $x = -1$. Next, find the equation of the line containing the median from A to $\overline{BC}$. The midpoint of $\overline{BC}$ is $\left(\frac{3+(-1)}{2}, \frac{-5+2}{2}\right)$, or $\left(1, -\frac{3}{2}\right)$. The slope from A to this midpoint is $\frac{-\frac{3}{2}-(-3)}{1-(-5)}$, or $\frac{1}{4}$. Using the definition of slope, the equation of the dashed line is $\frac{y-(-3)}{x-(-5)} = \frac{1}{4}$. Solving for y gives $y = \frac{1}{4}x - \frac{7}{4}$ as the equation of the line containing the median.



Finally, use substitution to solve this system.

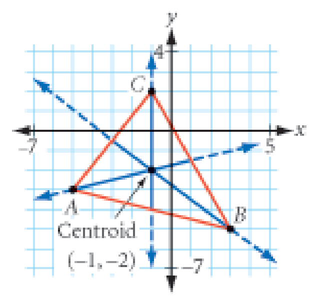
$$\begin{cases} y = \frac{1}{4}x - \frac{7}{4} \\ x = -1 \end{cases}$$

Equation of the line containing the median from A to $\overline{BC}$.
Equation of the line containing the median from C to $\overline{AB}$.

$y = \frac{1}{4}(-1) - \frac{7}{4}$ Substitute -1 for x in the first equation.

$y = -2$ Simplify.

The centroid is $(-1, -2)$. You can verify this result by writing the equation for the third median and making sure $(-1, -2)$ satisfies it.



b. The mean of the x-coordinates is

$$\frac{-5+3+(-1)}{3} = \frac{-3}{3} = -1.$$

The mean of the y-coordinates is

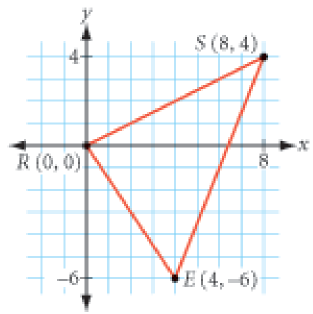
$$\frac{-3+(-5)+2}{3} = \frac{-6}{3} = -2.$$

Notice that these means give you the coordinates of the centroid: $(-1, -2)$.

You can generalize the findings from Example B to all triangles. The easiest way to find the coordinates of the centroid is to find the mean of the vertex coordinates.

Exercises

In Exercises 1 and 2, use $\triangle RES$ with vertices $R(0, 0)$, $E(4, -6)$, and $S(8, 4)$.

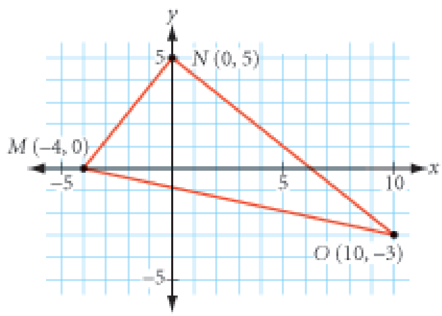


1. Find the equation of the line containing the median from R to $\overline{ES}$.

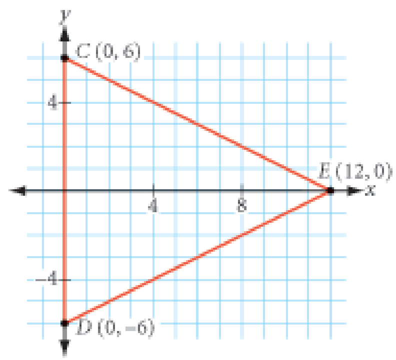
2. Find the equation of the line containing the altitude from E to $\overline{RS}$.

In Exercises 3 and 4, use algebra to find the coordinates of the centroid and the orthocenter for each triangle.

3. Right triangle MNO



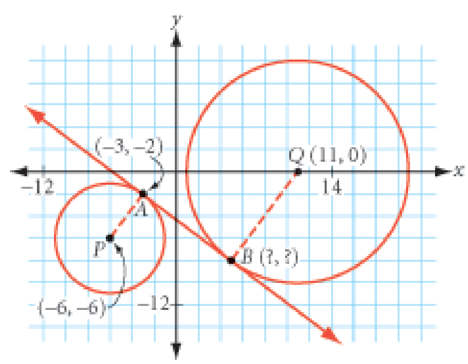
4. Isosceles triangle CDE



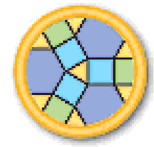
5. Find the coordinates of the centroid of the triangle formed by the x-axis, the y-axis, and the line $12x + 9y = 36$.

6. The three lines $8x + 3y = 12$, $6y - 7x = 24$, and $x + 9y + 33 = 0$ intersect to form a triangle. Find the coordinates of its centroid.

7. Circle P with center at $(-6, -6)$ and circle Q with center at $(11, 0)$ have a common internal tangent $\overleftrightarrow{AB}$. Find the coordinates of B if A has coordinates $(-3, -2)$.



CHAPTER 7 REVIEW



How is your memory? In this chapter you learned about rigid transformations in the plane—called isometries—and you revisited the principles of symmetry that you first learned in Chapter 0. You applied these concepts to create tessellations. Can you name the three rigid transformations? Can you describe how to compose transformations to make other transformations? How can you use reflections to improve your miniature-golf game? What types of symmetry do regular polygons have? What types of polygons will tile the plane? Review this chapter to be sure you can answer these questions.

Keymath.com Practice Your Skills

EXERCISES

For Exercises 1–12, identify each statement as true or false. For each false statement, sketch a counterexample or explain why it is false.

1. The two transformations in which the orientation (the order of points as you move clockwise) does not change are translation and rotation.

2. The two transformations in which the image has the opposite orientation from the original are reflection and glide reflection.

3. A translation by $\langle 5, 12 \rangle$ followed by a translation by $\langle -8, -6 \rangle$ is equivalent to a single translation by $\langle -3, 6 \rangle$.

4. A rotation of 140° followed by a rotation of 260° about the same point is equivalent to a single rotation of 40° about that point.

5. A reflection across a line followed by a second reflection across a parallel line that is 12 cm from the first is equivalent to a translation of 24 cm.

6. A regular n-gon has n reflectional symmetries and n rotational symmetries.

7. The only three regular polygons that create monohedral tessellations are equilateral triangles, squares, and regular pentagons.

8. Any triangle can create a monohedral tessellation.

9. Any quadrilateral can create a monohedral tessellation.

10. No pentagon can create a monohedral tessellation.

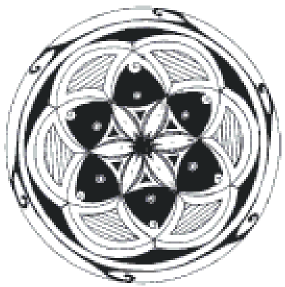
11. No hexagon can create a monohedral tessellation.

12. There are at least three times as many true statements as false statements in Exercises 1–12.

King by Minnie Evans (1892–1987)

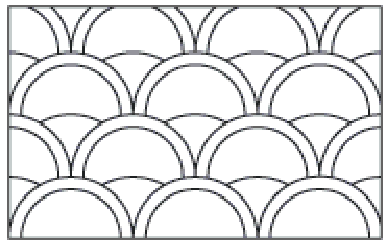
In Exercises 13–15, identify the type or types of symmetry, including the number of symmetries, in each design. For Exercise 15, describe how you can move candles on the menorah to make the colors symmetrical, too.

13.



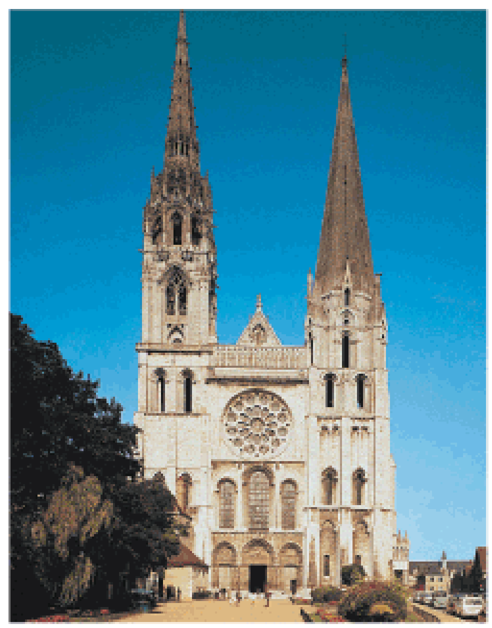
Mandala,
Gary Chen, geometry student

14.



15.

16. The façade of Chartres Cathedral in France does not have reflectional symmetry. Why not? Sketch the portion of the façade that does have bilateral symmetry.

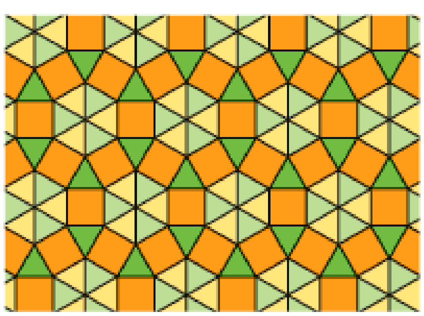


17. Find or create a logo that has reflectional symmetry. Sketch the logo and its line or lines of reflectional symmetry.

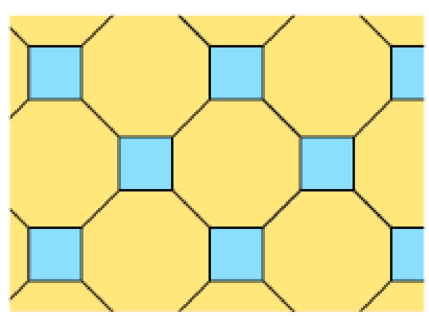
18. Find or create a logo that has rotational symmetry, but not reflectional symmetry. Sketch it.

In Exercises 19 and 20, classify the tessellation and give the vertex arrangement.

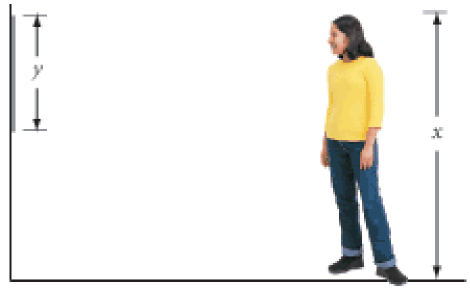
19.



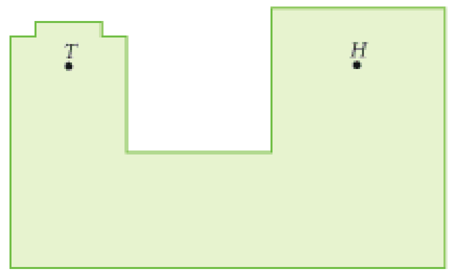
20.



21. Experiment with a mirror to find the smallest vertical portion (y) in which you can still see your full height (x). How does y compare to x? Can you explain, with the help of a diagram and what you know about reflections, why a "full-length mirror need not be as tall as you?

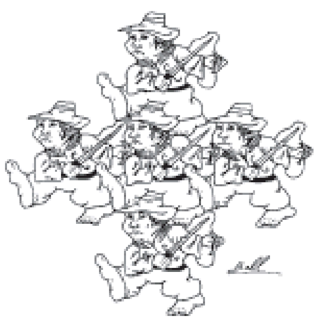


22. Miniature-golf pro Sandy Trapp wishes to impress her fans with a hole in one on the very first try. How should she hit the ball at T to achieve this feat? Explain.



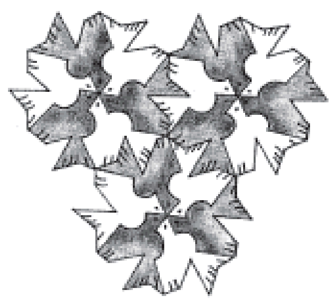
In Exercises 23–25, identify the shape of the tessellation grid and a possible method that the student used to create each tessellation.

23.



Perian Warriors, Robert Bell

24.



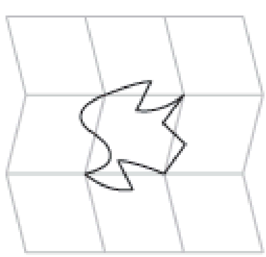
Doves, Serene Tam

25.

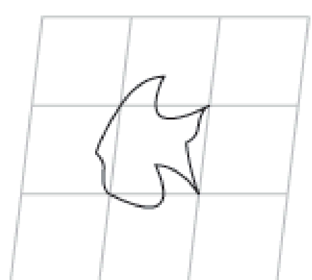
Sightings,
Peter Chua and Monica Grant

In Exercises 26 and 27, copy the figure and grid onto patty paper. Determine whether or not you can use the figure to create a tessellation on the grid. Explain your reasoning.

26.

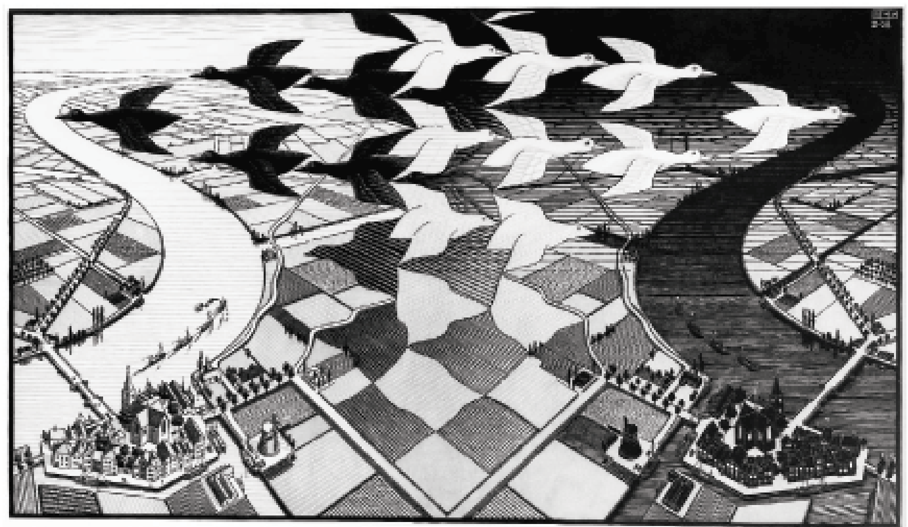


27.



28. In his woodcut *Day and Night,* Escher gradually changes the shape of the patches of farmland into black and white birds. The birds are flying in opposite directions, so they appear to be glide reflections of each other. But notice that the tails of the white birds curve down, while the tails of the black birds curve up. So, on closer inspection, it's clear that this is not a glide-reflection tiling at all!

When two birds are taken together as one tile (a 2-motif tile), they create a translation tessellation. Use patty paper to find the 2-motif tile.

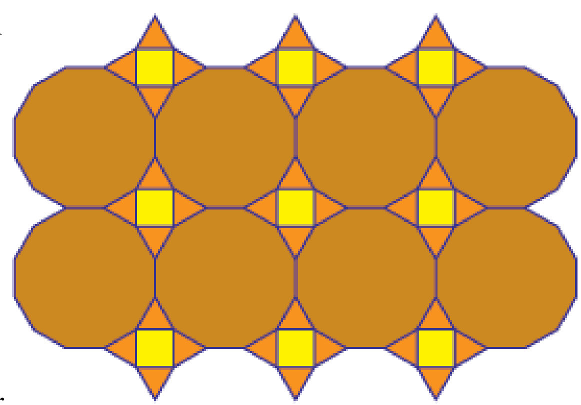


Day and Night, M. C. Escher, 1938

TAKE ANOTHER LOOK

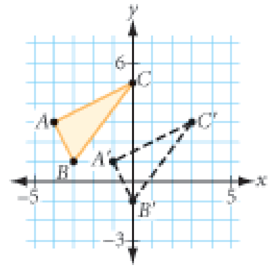
1. Assuming there are no other balls on a pool table, where should a player aim so that a randomly placed cue ball bounces off exactly three different cushions before returning to its original spot? How many different solutions can you find?

2. There are 20 different 2-uniform tilings. Seven of them are shown in this chapter: on page 391, Exercises 6–8 on page 392, Exercise 13 on page 402, Exercise 19 on page 417, and the one shown here. Use pattern blocks or geometry software to find at least five more 2-uniform tilings.



3. There are many books and websites devoted to the tessellation art of M. C. Escher. Make copies of three different Escher tessellation designs not found in this textbook. One should be a translation-type tessellation, one a rotation-type tessellation, and one a glide reflection-type tessellation. Label them with their tessellation type and use patty paper to locate the unit tessellation shape for each design.

4. Matrices can be used to translate polygons on the coordinate plane. For example, to translate $\triangle ABC$ by the vector $\langle 3, -2\rangle$, you can add a translation matrix to a matrix of the triangle's vertices. The result is an image matrix that gives the coordinates of the image's vertices.

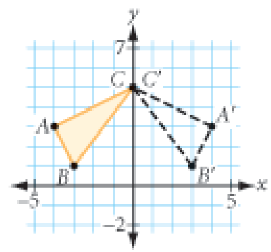


[triangle matrix] + [translation matrix] = [image matrix]

$$\begin{bmatrix} -4 & -3 & 0 \\ 3 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 \\ -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 \\ 1 & -1 & 3 \end{bmatrix}$$

Pentagon *ANGIE* has vertices $A(0, 0)$, $N(6, 0)$, $G(7, 3)$, $I(2, 7)$, $E(0, 5)$. Translate *ANGIE* by the vector $\langle -5, -3\rangle$. Show the matrix addition that results in the image matrix for pentagon $A'N'G'I'E'$, and verify your results by graphing the original pentagon and its image, either on paper or a graphing calculator.

5. Matrices can also be used to reflect or rotate polygons on the coordinate plane. For example, to reflect $\triangle ABC$ from Exercise 4 across the *y*-axis, you can multiply the triangle matrix by the transformation matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.



$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & -3 & 0 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 1 & 5 \end{bmatrix}$$

Multiply the triangle matrix for $\triangle ABC$ by each of these transformation matrices. You can use a graphing calculator. Graph $\triangle ABC$ and its image, $\triangle A'B'C'$, on the same coordinate system and describe the resulting transformation.

a. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **b.** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **e.** $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Assessing What You've Learned

UPDATE YOUR PORTFOLIO Choose one of the tessellations you did in this chapter and add it to your portfolio. Describe why you chose it and explain the transformations you used and the types of symmetry it has.

ORGANIZE YOUR NOTEBOOK Review your notebook to be sure it's complete and well organized. Are all the types of transformation and symmetry included in your definition list or conjecture list? Write a one-page chapter summary.

WRITE IN YOUR JOURNAL This chapter emphasizes applying geometry to create art. Write about connections you see between geometry and art. Does creating geometric art give you a greater appreciation for either art or geometry? Explain.

PERFORMANCE ASSESSMENT While a classmate, a friend, a family member, or a teacher observes, carry out one of the investigations from this chapter. Explain what you're doing at each step, including how you arrive at the conjecture.

GIVE A PRESENTATION Give a presentation about one of the investigations or projects you did or about one of the tessellations you created.

