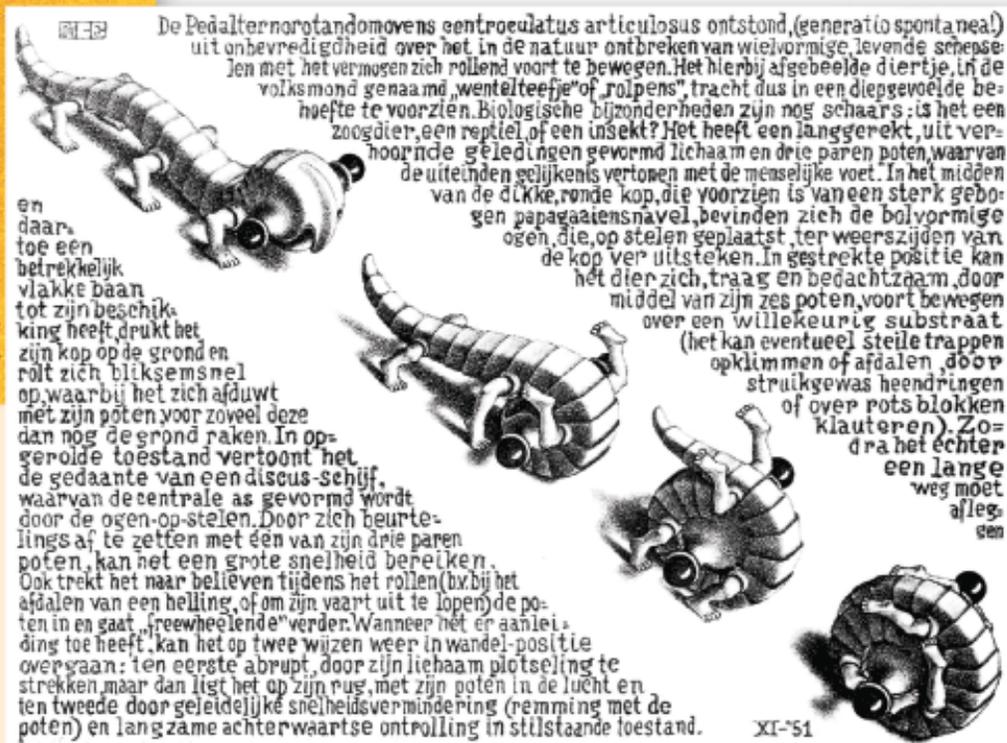


Discovering and Proving Circle Properties



*I am the only one who can judge
how far I constantly remain below
the quality I would like to attain.*

M.C. ESCHER

Curl-Up, M.C. Escher, 1951
©2002 Cordon Art B.V.–Baarn–Holland.
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OBJECTIVES

In this chapter you will

- discover properties of tangent lines
- learn relationships among chords, arcs, and angles
- learn how to calculate the length of an arc
- prove circle conjectures

Tangent Properties

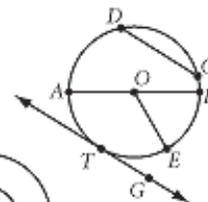
Let's review some basic terms from Chapter 1 before you begin discovering the properties of circles. You should be able to identify the terms below.

Match the figures at the right with the terms at the left.

*We are, all of us, alone
Though not uncommon
In our singularity.
Touching,
We become tangent to
Circles of common experience,
Co-incident,
Defining in collective tangency
Circles
Reciprocal in their subtle
Redefinition of us.
In tangency
We are never less alone,
But no longer
Only.
GENE MATTINGLY*

1. Congruent circles
2. Concentric circles
3. Radius
4. Chord
5. Diameter
6. Tangent
7. Central angle
8. Minor arc
9. Major arc
10. Semicircle

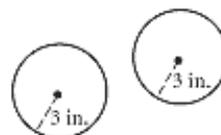
- A. \overline{DC}
- B. \overline{TG}
- C. \overline{OE}
- D. \overline{AB}



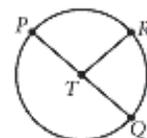
E.



F.



- G. \overline{RQ}
- H. \overline{PRQ}
- I. \overline{PQR}
- J. $\angle PTR$



Check your answers: (1.F, 2.E, 3.C, 4.A, 5.D, 6.B, 7.J, 8.G, 9.I, 10.H)



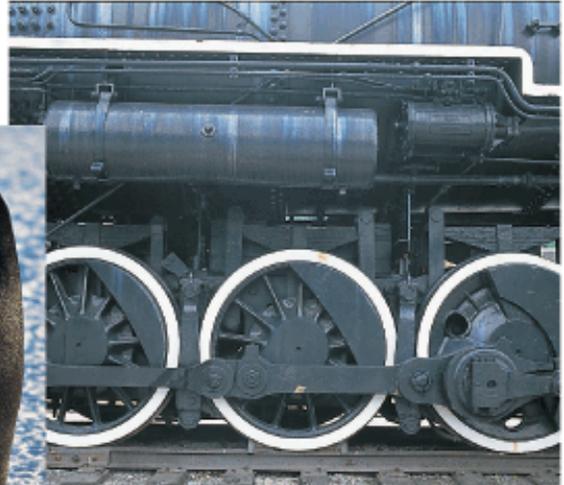
Double Splash Evidence, part of modern California artist Gerrit Greve's *Water Series*, uses brushstrokes to produce an impression of concentric ripples in water.



Can you find parts of the water wheel that match the circle terms above?

In this lesson you will investigate the relationship between a tangent line to a circle and the radius of the circle, and between two tangent segments to a common point outside the circle.

Rails act as tangent lines to the wheels of a train. Each wheel of a train theoretically touches only one point on the rail. The point where the rail and the wheel meet is a point of tangency. Why can't a train wheel touch more than one point at a time on the rail? How is the radius of the wheel to the point of tangency related to the rail? Let's investigate.



The rail is tangent to the wheels of the train. The penguins' heads are tangent to each other.

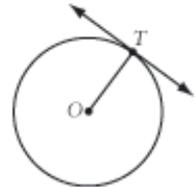


Investigation 1 Going Off on a Tangent

You will need

- a compass
- a straightedge

In this investigation you will discover the relationship between a tangent line and the radius drawn to the point of tangency.



- Step 1 Construct a large circle. Label the center O .
- Step 2 Using your straightedge, draw a line that appears to touch the circle at only one point. Label the point T . Construct \overline{OT} .
- Step 3 Use your protractor to measure the angles at T . What can you conclude about the radius \overline{OT} and the tangent line at T ?
- Step 4 Share your results with your group. Then copy and complete the conjecture.

Tangent Conjecture

C-53

A tangent to a circle ? the radius drawn to the point of tangency.

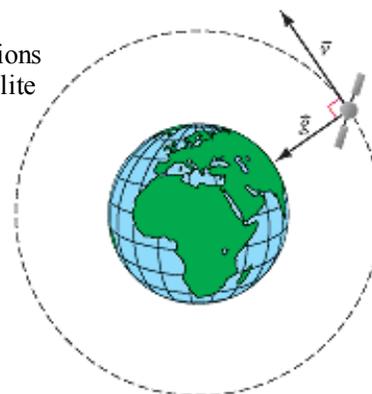
Technology CONNECTION

At right is the Mir space station and the space shuttle *Atlantis* in orbit in 1995. According to the United States Space Command, there are over 8,000 objects larger than a softball circling Earth at speeds of over 18,000 miles per hour! If gravity were suddenly “turned off” somehow, these objects would suddenly travel off into space on a straight line tangent to their orbits, and not continue in a curved path.



You can also show that the converse of the Tangent Conjecture is true. If you construct a line perpendicular to a radius at the point where it touches the circle, the line will be tangent to the circle.

The Tangent Conjecture has important applications related to circular motion. For example, a satellite maintains its velocity in a direction tangent to its circular orbit. This velocity vector is perpendicular to the force of gravity, which keeps the satellite in orbit.



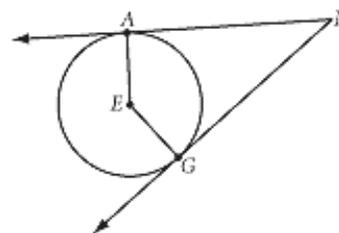
Investigation 2 Tangent Segments

You will need

- a compass
- a straightedge

In this investigation you will discover something about the lengths of segments tangent to a circle from a point outside the circle.

- Step 1 Construct a circle. Label the center E .
- Step 2 Choose a point outside the circle and label it N .
- Step 3 Draw two lines through point N tangent to the circle. Mark the points where these lines appear to touch the circle and label them A and G .
- Step 4 Use your compass to compare segments NA and NG . Segments such as these are called **tangent segments**.
- Step 5 Share your results with your group. Copy and complete the conjecture.

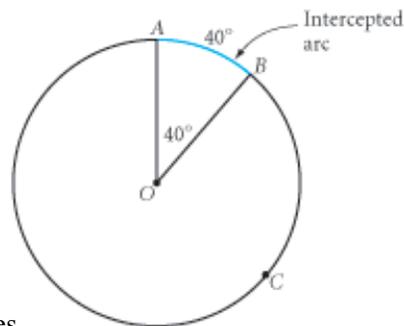


Tangent Segments Conjecture

C-54

Tangent segments to a circle from a point outside the circle are .

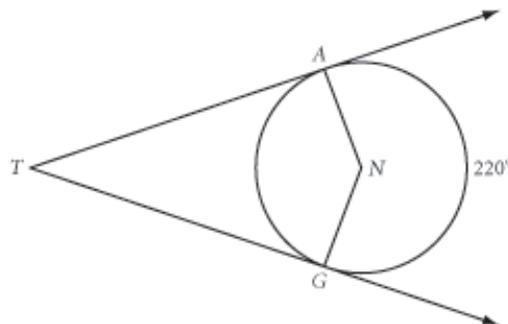
In the figure at right, the central angle, $\angle BOA$, determines the minor arc, \overline{AB} . $\angle BOA$ is said to **intercept** \overline{AB} because the arc is within the angle. The measure of a minor arc is defined as the measure of its central angle, so $m\overline{AB} = 40^\circ$. The measure of a major arc is the reflex measure of $\angle BOA$, or 360° minus the measure of the minor arc, so $m\overline{BCA} = 320^\circ$.



Let's look at an example involving arc measures and tangent segments.

EXAMPLE

In the figure at right, \overline{TA} and \overline{TG} are both tangent to circle N . If the major arc formed by the two tangents measures 220° , find the measure of $\angle T$.



Solution

The minor arc intercepted by $\angle N$ measures $360^\circ - 220^\circ$, or 140° . Thus, $m\angle N = 140^\circ$. By the Tangent Conjecture, both $\angle A$ and $\angle G$ must be right angles, and by the Quadrilateral Sum Conjecture, the sum of the angles in $TANG$ is 360° .

So, $m\angle T + 90^\circ + 140^\circ + 90^\circ = 360^\circ$, which means that $m\angle T = 40^\circ$.

Tangent circles are two circles that are tangent to the same line at the same point. They can be **internally tangent** or **externally tangent**, as shown.



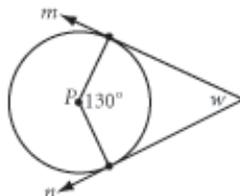
Externally tangent circles

Internally tangent circles

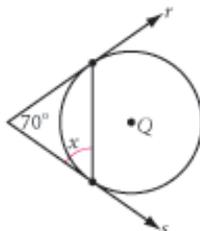


EXERCISES

1. Rays m and n are tangent to circle P . $w = ?$



2. Rays r and s are tangent to circle Q . $x = ?$



You will need

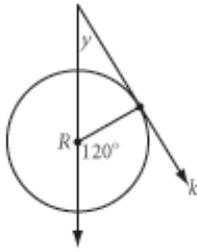


Construction tools
for Exercises 8–12, 15, and 20

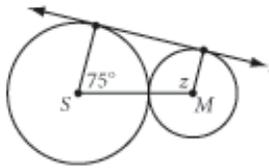


Geometry software
for Exercise 13

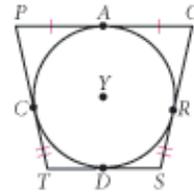
3. Ray k is tangent to circle R .
 $y = \underline{\hspace{1cm}}$



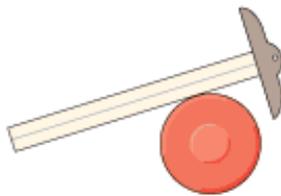
4. Line t is tangent to both
 tangent circles. $z = \underline{\hspace{1cm}}$



5. Quadrilateral $POST$ is
 circumscribed about circle Y .
 $OR = 13$ in. and $ST = 12$ in.
 Find the perimeter of $POST$.



6. Pam participates in the hammer-throw event. She swings a 16-lb ball at arm's length, about eye-level. Then she releases the ball at the precise moment when the ball will travel in a straight line toward the target area. Draw an overhead view that shows the ball's circular path, her arms at the moment she releases it, and the ball's straight path toward the target area.
7. Explain how you could use only a T-square, like the one shown, to find the center of a Frisbee.



Pam Dukes competes in the hammer-throw event.

Construction For Exercises 8–12, first make a sketch of what you are trying to construct and label it. Then use the segments below, with lengths r , s , and t .



8. Construct a circle with radius r . Mark a point on the circle. Construct a tangent through this point.
9. Construct a circle with radius t . Choose three points on the circle that divide it into three minor arcs and label points X , Y , and Z . Construct a triangle that is circumscribed about the circle and tangent at points X , Y , and Z .
10. Construct two congruent, externally tangent circles with radius s . Then construct a third circle that is both congruent and externally tangent to the two circles.
11. Construct two internally tangent circles with radii r and t .
12. Construct a third circle with radius s that is externally tangent to both the circles you constructed in Exercise 11.
13. **Technology** Use geometry software to construct a circle. Label three points on the circle and construct tangents through them. Drag the three points and write your observations about where the tangent lines intersect and the figures they form. What happens if two of the three points are on opposite ends of a diameter? What happens if the three points are on the same semicircle?

14. Find real-world examples (different from the examples shown below) of two internally tangent circles and of two externally tangent circles. Either sketch the examples or make photocopies from a book or a magazine for your notebook.



The teeth in the gears shown extend from circles that are externally tangent.

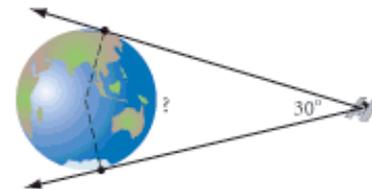


This astronomical clock in Prague, Czech Republic, has one pair of internally tangent circles. What other circle relationships can you find in the clock photo?

15. **Construction** In Taoist philosophy, all things are governed by one of two natural principles, yin and yang. Yin represents the earth, characterized by darkness, cold, or wetness. Yang represents the heavens, characterized by light, heat, or dryness. The two principles, when balanced, combine to produce the harmony of nature. The symbol for the balance of yin and yang is shown at right. Construct the yin-and-yang symbol. Start with one large circle. Then construct two circles with half the diameter that are internally tangent to the large circle and externally tangent to each other. Finally, construct small circles that are concentric to the two inside circles. Shade or color your construction. 



16. A satellite in geostationary orbit remains above the same point on Earth's surface even as Earth turns. If such a satellite has a 30° view of the equator, what percentage of the equator is observable from the satellite? 



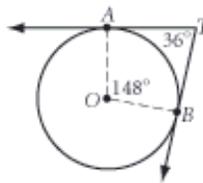
Astronomy

CONNECTION

Tangent lines can help you locate where a solar eclipse will occur on the surface of Earth. The diagram at right shows how rays tangent to both the sun and moon will determine the boundaries of regions that experience total shadow, called the umbra, and partial shadow, called the penumbra. This diagram is not drawn to scale. The diameter of the sun is roughly 400 times larger than that of the moon, but the sun is also about 400 times farther away from Earth than the moon, making them appear roughly the same size in the sky.



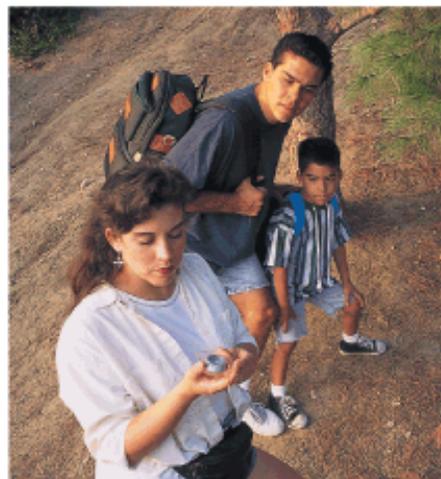
17. **Developing Proof** \overrightarrow{TA} and \overrightarrow{TB} are tangent to circle O . What's wrong with this picture?



Review

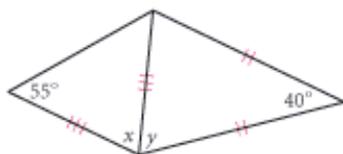
18. Identify each quadrilateral from the given characteristics.
- Diagonals are perpendicular and bisect each other.
 - Diagonals are congruent and bisect each other, but it is not a square.
 - Only one diagonal is the perpendicular bisector of the other diagonal.
 - Diagonals bisect each other.

19. A family hikes from their camp on a bearing of 15° . (A **bearing** is an angle measured clockwise from the north, so a bearing of 15° is 15° east of north.) They hike 6 km and then stop for a swim in a lake. Then they continue their hike on a new bearing of 117° . After another 9 km, they meet their friends. What is the measure of the angle between the path they took to arrive at the lake and the path they took to leave the lake? **h**



20. **Construction** Use a protractor and a centimeter ruler to make a careful drawing of the route the family in Exercise 20 traveled to meet their friends. Let 1 cm represent 1 km. To the nearest tenth of a kilometer, how far are they from their first camp?

21. Explain why x equals y .



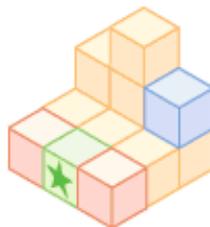
22. What is the probability of randomly selecting three points from the 3-by-3 grid below that form the vertices of a right triangle?



IMPROVING YOUR VISUAL THINKING SKILLS

Colored Cubes

Sketch the solid shown, but with the red cubes removed and the blue cube moved to cover the starred face of the green cube.

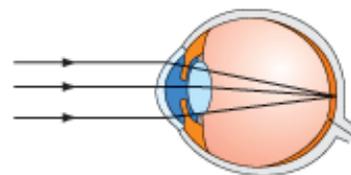


Chord Properties

*You will do foolish things,
but do them with enthusiasm.*
SIDONIE GABRIELLA COLETTE

In the last lesson you discovered some properties of a tangent, a line that intersects the circle only once. In this lesson you will investigate properties of a chord, a line segment whose endpoints lie on the circle.

In a person with correct vision, light rays from distant objects are focused to a point on the retina. If the eye represents a circle, then the path of the light from the lens to the retina represents a chord. The angle formed by two of these chords to the same point on the retina represents an inscribed angle.



First you will define two types of angles in a circle.



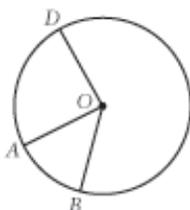
Investigation 1

Defining Angles in a Circle

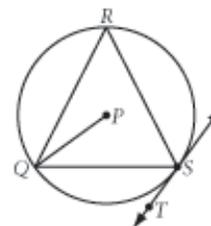
Write a good definition of each boldfaced term. Discuss your definitions with others in your group. Agree on a common set of definitions as a class and add them to your definition list. In your notebook, draw and label a figure to illustrate each term.

Step 1

Central Angle



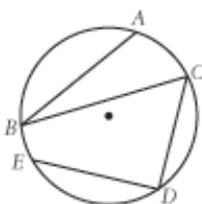
$\angle AOB$, $\angle DOA$, and $\angle DOB$ are central angles of circle O .



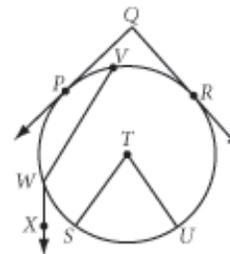
$\angle PQR$, $\angle PQS$, $\angle RST$, $\angle QST$, and $\angle QSR$ are not central angles of circle P .

Step 2

Inscribed Angle



$\angle ABC$, $\angle BCD$, and $\angle CDE$ are inscribed angles.



$\angle PQR$, $\angle STU$, and $\angle VWX$ are not inscribed angles.



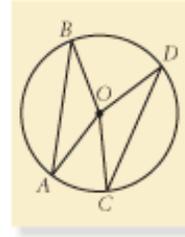
Investigation 2

Chords and Their Central Angles

You will need

- a compass
- a straightedge
- a protractor
- patty paper (optional)

Next you will discover some properties of chords and central angles. You will also see a relationship between chords and arcs.



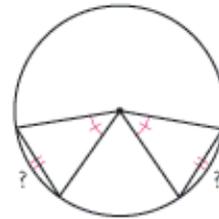
- Step 1 Construct a large circle. Label the center O . Construct two congruent chords in your circle. Label the chords \overline{AB} and \overline{CD} , then construct radii \overline{OA} , \overline{OB} , \overline{OC} , and \overline{OD} .
- Step 2 With your protractor, measure $\angle BOA$ and $\angle COD$. How do they compare? Share your results with others in your group. Then copy and complete the conjecture.

Chord Central Angles Conjecture

C-55

If two chords in a circle are congruent, then they determine two central angles that are ?

- Step 3 How can you fold your circle construction to check the conjecture?
- Step 4 Recall that the measure of an arc is defined as the measure of its central angle. If two central angles are congruent, their intercepted arcs must be congruent. Combine this fact with the Chord Central Angles Conjecture to complete the next conjecture.



Chord Arcs Conjecture

C-56

If two chords in a circle are congruent, then their ? are congruent.

"Pull the cord?! Don't I need to construct it first?"





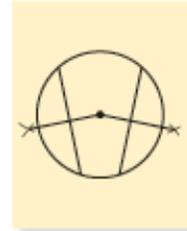
Investigation 3

Chords and the Center of the Circle

You will need

- a compass
- a straightedge
- patty paper (optional)

In this investigation you will discover relationships about a chord and the center of its circle.



- Step 1 Construct a large circle and mark the center. Construct two nonparallel congruent chords. Then construct the perpendiculars from the center to each chord.
- Step 2 How does the perpendicular from the center of a circle to a chord divide the chord? Copy and complete the conjecture.

Perpendicular to a Chord Conjecture

C-57

The perpendicular from the center of a circle to a chord is the ? of the chord.

Let's continue this investigation to discover a relationship between the length of congruent chords and their distances from the center of the circle.

- Step 3 Compare the distances (measured along the perpendicular) from the center to the chords. Are the results the same if you change the size of the circle and the length of the chords? State your observations as your next conjecture.

Chord Distance to Center Conjecture

C-58

Two congruent chords in a circle are ? from the center of the circle.



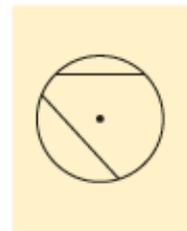
Investigation 4

Perpendicular Bisector of a Chord

You will need

- a compass
- a straightedge
- patty paper (optional)

Next, you will discover a property of perpendicular bisectors of chords.



- Step 1 Construct a large circle and mark the center. Construct two nonparallel chords that are not diameters. Then construct the perpendicular bisector of each chord and extend the bisectors until they intersect.

Step 2 | What do you notice about the point of intersection? Compare your results with the results of others near you. Copy and complete the conjecture.

Perpendicular Bisector of a Chord Conjecture

C-59

The perpendicular bisector of a chord .



keymath.com/DG

▶ For interactive versions of these investigations, see the **Dynamic Geometry Exploration** Chord Properties at www.keymath.com/DG.

With the perpendicular bisector of a chord, you can find the center of any circle, and therefore the vertex of the central angle to any arc. All you have to do is construct the perpendicular bisectors of nonparallel chords.



EXERCISES

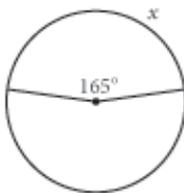
You will need



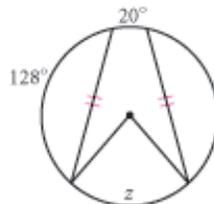
Construction tools
for Exercises 16–19

Solve Exercises 1–10. State which conjectures or definitions you used.

1. $x = \underline{\quad?}$



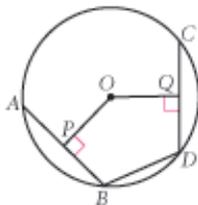
2. $z = \underline{\quad?}$



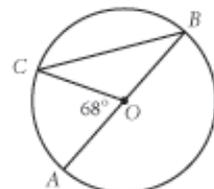
3. $w = \underline{\quad?}$



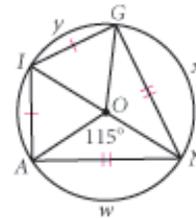
4. $AB = CD$
 $PO = 8$ cm
 $OQ = \underline{\quad?}$



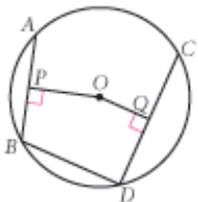
5. \overline{AB} is a diameter. Find $m\widehat{AC}$ and $m\angle B$.



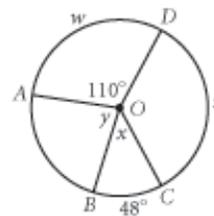
6. $GIAN$ is a kite. Find w , x , and y .



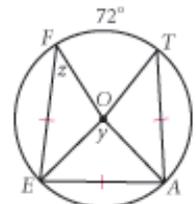
7. $AB = 6$ cm $OP = 4$ cm
 $CD = 8$ cm $OQ = 3$ cm
 $BD = 6$ cm
What is the perimeter of $OPBDQ$?



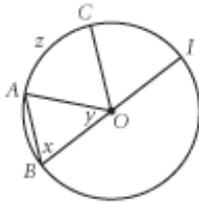
8. $m\widehat{AC} = 130^\circ$
Find w , x , y , and z .



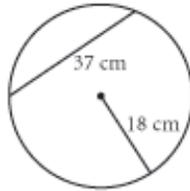
9. $x = \underline{\quad?}$
 $y = \underline{\quad?}$ (h)
 $z = \underline{\quad?}$



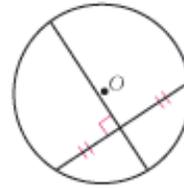
10. $\overline{AB} \parallel \overline{CO}$, $m\widehat{CI} = 66^\circ$
Find x , y , and z .



11. **Developing Proof** What's wrong with this picture?



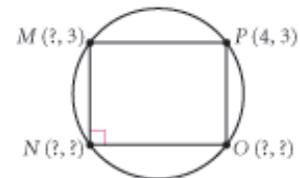
12. **Developing Proof** What's wrong with this picture?



13. Draw a circle and two chords of unequal length. Which is closer to the center of the circle, the longer chord or the shorter chord? Explain.

14. Draw two circles with different radii. In each circle, draw a chord so that the chords have the same length. Draw the central angle determined by each chord. Which central angle is larger? Explain.

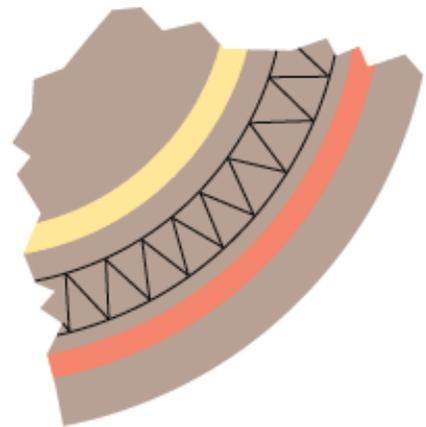
15. Polygon $MNOP$ is a rectangle inscribed in a circle centered at the origin. Find the coordinates of points M , N , and O .



16. **Construction** Construct a triangle. Using the sides of the triangle as chords, construct a circle passing through all three vertices. Explain. Why does this seem familiar?

17. **Construction** Trace a circle onto a blank sheet of paper without using your compass. Locate the center of the circle using a compass and straightedge. Trace another circle onto patty paper and find the center by folding.

18. **Construction** Adventurer Dakota Davis digs up a piece of a circular ceramic plate. Suppose he believes that some ancient plates with this particular design have a diameter of 15 cm. He wants to calculate the diameter of the original plate to see if the piece he found is part of such a plate. He has only this piece of the circular plate, shown at right, to make his calculations. Trace the outer edge of the plate onto a sheet of paper. Help him find the diameter.



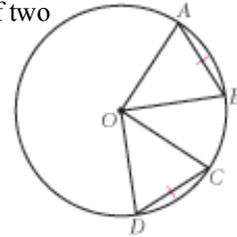
19. **Construction** The satellite photo at right shows only a portion of a lunar crater. How can cartographers use the photo to find its center? Trace the crater and locate its center. Using the scale shown, find its radius. To learn more about satellite photos, go to www.keymath.com/DG.



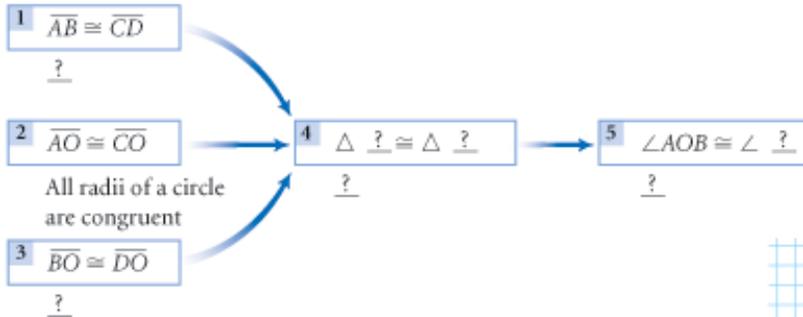
20. **Developing Proof** Complete the flowchart proof shown, which proves that if two chords of a circle are congruent, then they determine two congruent central angles.

Given: Circle O with chords $\overline{AB} \cong \overline{CD}$

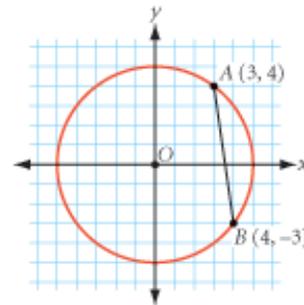
Show: $\angle AOB \cong \angle COD$



Flowchart Proof

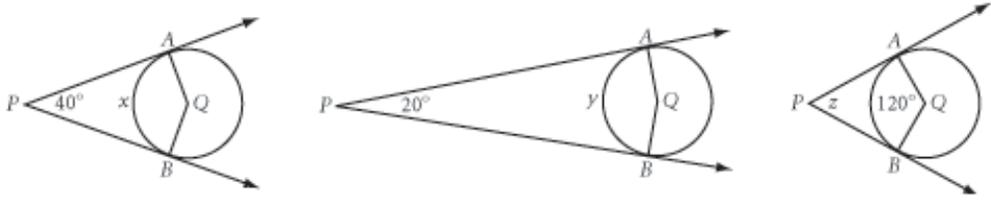


21. Circle O has center $(0, 0)$ and passes through points $A(3, 4)$ and $B(4, -3)$. Find an equation to show that the perpendicular bisector of \overline{AB} passes through the center of the circle. Explain your reasoning.



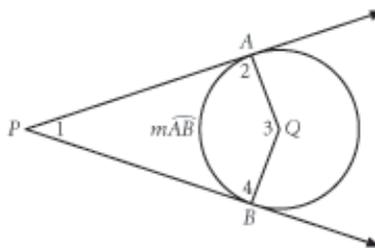
Review

22. **Developing Proof** Identify each of these statements as true or false. If the statement is true, explain why. If it is false, give a counterexample.
- If the diagonals of a quadrilateral are congruent, but only one is the perpendicular bisector of the other, then the quadrilateral is a kite.
 - If the quadrilateral has exactly one line of reflectional symmetry, then the quadrilateral is a kite.
 - If the diagonals of a quadrilateral are congruent and bisect each other, then it is a square.
23. **Mini-Investigation** Use what you learned in the last lesson about the angle formed by a tangent and a radius to find the missing arc measure or angle measure in each diagram. Examine these cases to find a relationship between the measure of the angle formed by two tangents to a circle, $\angle P$, and the measure of the intercepted arc, \overline{AB} . Then copy and complete the conjecture below.

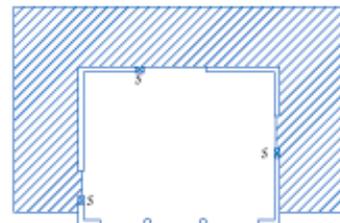


Conjecture: The measure of the angle formed by two intersecting tangents to a circle is $\frac{1}{2}$ (Intersecting Tangents Conjecture).

24. **Developing Proof** Given that \overline{PA} and \overline{PB} are both tangent to circle Q in the diagram at right, prove the conjecture you made in the last exercise.

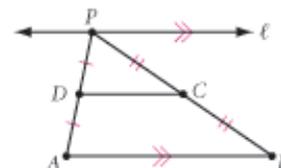


25. Rachel and Yulia are building an art studio above their back bedroom. There will be doors on three sides leading to a small deck that surrounds the studio. They need to place an electrical junction box in the ceiling of the studio so that it is equidistant from the three light switches shown by the doors. Copy the diagram of the room and find the location of the junction box. 



26. What will the units digit be when you evaluate 3^{23} ? 
27. A small light-wing aircraft has made an emergency landing in a remote portion of a wildlife refuge and is sending out radio signals for help. Ranger Station Alpha receives the signal on a bearing of 38° and Station Beta receives the signal on a bearing of 312° . (Recall that a bearing is an angle measured clockwise from the north.) Stations Alpha and Beta are 8.2 miles apart, and Station Beta is on a bearing of 72° from Station Alpha. Which station is closer to the downed aircraft? Explain your reasoning. 

28. Consider the figure at right with line $\ell \parallel \overline{AB}$. As P moves from left to right along line ℓ , which of these values does not remain constant?
- The length of \overline{DC}
 - The distance from D to \overline{AB}
 - The ratio $DC : AB$
 - The perimeter of $\triangle ABP$
 - None of the above



IMPROVING YOUR ALGEBRA SKILLS

Algebraic Sequences II

Find the next two terms of each algebraic pattern.

- $x^6, 6x^5y, 15x^4y^2, 20x^3y^3, 15x^2y^4, \underline{\quad}, \underline{\quad}$
- $x^7, 7x^6y, 21x^5y^2, 35x^4y^3, 35x^3y^4, 21x^2y^5, \underline{\quad}, \underline{\quad}$
- $x^8, 8x^7y, 28x^6y^2, 56x^5y^3, 70x^4y^4, 56x^3y^5, 28x^2y^6, \underline{\quad}, \underline{\quad}$
- $x^9, 9x^8y, 36x^7y^2, 84x^6y^3, 126x^5y^4, 126x^4y^5, 84x^3y^6, \underline{\quad}, \underline{\quad}, \underline{\quad}$



Arcs and Angles

Many arches that you see in structures are semicircular, but Chinese builders long ago discovered that arches don't have to have this shape. The Zhaozhou bridge, shown below, was completed in 605 C.E. It is the world's first stone arched bridge in the shape of a minor arc, predating other minor-arc arches by about 800 years.

In this lesson you'll discover properties of arcs and the angles associated with them.

Learning by experience is good, but in the case of mushrooms and toadstools, hearsay evidence is better.

ANONYMOUS

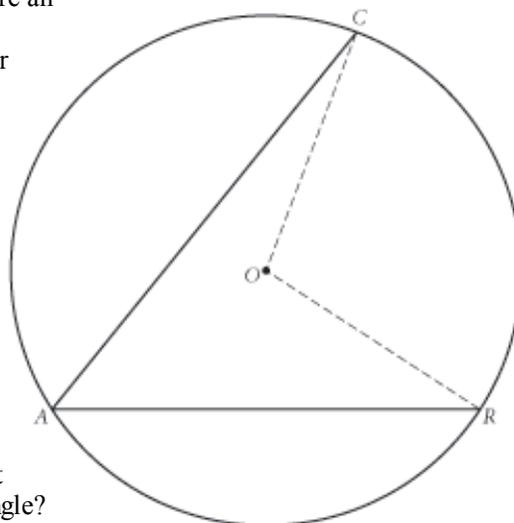


Investigation 1 Inscribed Angle Properties

You will need

- a compass
- a straightedge
- a protractor

In this investigation you will compare an inscribed angle and a central angle, both inscribed in the same arc. Refer to the diagram of circle O , with central angle COR and inscribed angle CAR .



- Step 1 Measure $\angle COR$ with your protractor to find $m\overline{CR}$, the intercepted arc. Measure $\angle CAR$. How does $m\angle CAR$ compare with $m\overline{CR}$?
- Step 2 Construct a circle of your own with an inscribed angle. Draw and measure the central angle that intercepts the same arc. What is the measure of the inscribed angle? How do the two measures compare?
- Step 3 Share your results with others near you. Copy and complete the conjecture.

Inscribed Angle Conjecture

C-60

The measure of an angle inscribed in a circle ?.



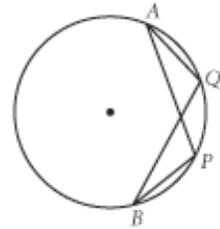
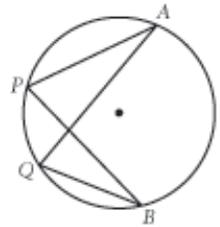
Investigation 2

Inscribed Angles Intercepting the Same Arc

You will need

- a compass
- a straightedge
- a protractor

- Next, let's consider two inscribed angles that intercept the same arc. In the figure at right, $\angle AQB$ and $\angle APB$ both intercept \overline{AB} . Angles AQB and APB are both inscribed in \overline{APB} .
- Step 1 Construct a large circle. Select two points on the circle. Label them A and B . Select a point P on the major arc and construct inscribed angle APB . With your protractor, measure $\angle APB$.
- Step 2 Select another point Q on \overline{APB} and construct inscribed angle AQB . Measure $\angle AQB$.
- Step 3 How does $m\angle AQB$ compare with $m\angle APB$?
- Step 4 Repeat Steps 1–3 with points P and Q selected on minor arc AB . Compare results with your group. Then copy and complete the conjecture.



Inscribed Angles Intercepting Arcs Conjecture

C-61

Inscribed angles that intercept the same arc .



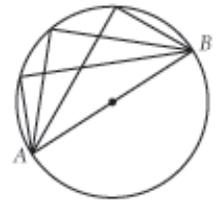
Investigation 3

Angles Inscribed in a Semicircle

You will need

- a compass
- a straightedge
- a protractor

- Next, you will investigate a property of angles inscribed in semicircles. This will lead you to a third important conjecture about inscribed angles.
- Step 1 Construct a large circle. Construct a diameter \overline{AB} . Inscribe three angles in the same semicircle. Make sure the sides of each angle pass through A and B .
- Step 2 Measure each angle with your protractor. What do you notice? Compare your results with the results of others and make a conjecture.



Angles Inscribed in a Semicircle Conjecture

C-62

Angles inscribed in a semicircle .

Now you will discover a property of the angles of a quadrilateral inscribed in a circle.



Investigation 4

Cyclic Quadrilaterals

You will need

- a compass
- a straightedge
- a protractor

A quadrilateral inscribed in a circle is called a **cyclic quadrilateral**. Each of its angles is inscribed in the circle, and each of its sides is a chord of the circle.



- Step 1 Construct a large circle. Construct a cyclic quadrilateral by connecting four points anywhere on the circle.
- Step 2 Measure each of the four inscribed angles. Write the measure in each angle. Look carefully at the sums of various angles. Share your observations with students near you. Then copy and complete the conjecture.

Cyclic Quadrilateral Conjecture

C-63

The ? angles of a cyclic quadrilateral are ?.



Investigation 5

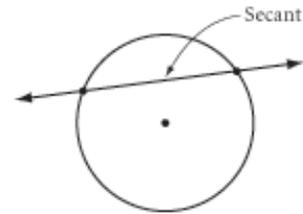
Arcs by Parallel Lines

You will need

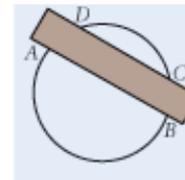
- patty paper
- a compass
- a double-edged straightedge

Next, you will investigate arcs formed by parallel lines that intersect a circle.

A line that intersects a circle in two points is called a **secant**. A secant contains a chord of the circle, and passes through the interior of a circle, while a tangent line does not. Note that a secant is a line while a chord is a segment.



- Step 1 On a piece of patty paper, construct a large circle. Lay your straightedge across the circle so that its parallel edges pass through the circle. Draw secants \overline{AB} and \overline{DC} along both edges of the straightedge.
- Step 2 Fold your patty paper to compare \overline{AD} and \overline{BC} . What can you say about \overline{AD} and \overline{BC} ?
- Step 3 Repeat Steps 1 and 2, using either lined paper or another object with parallel edges to construct different parallel secants. Share your results with other students. Then copy and complete the conjecture.



Parallel Lines Intercepted Arcs Conjecture

C-64

Parallel lines intercept ? arcs on a circle.

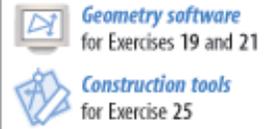
Review these conjectures and ask yourself which quadrilaterals can be inscribed in a circle. Can any parallelogram be a cyclic quadrilateral? If two sides of a cyclic quadrilateral are parallel, then what kind of quadrilateral will it be?



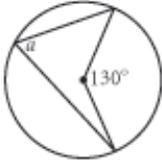
EXERCISES

You will need

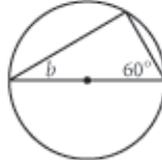
Use your new conjectures to solve Exercises 1–17.



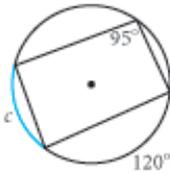
1. $a = ?$



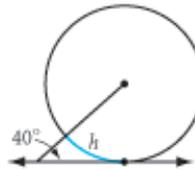
2. $b = ?$



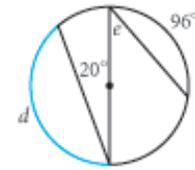
3. $c = ?$ (h)



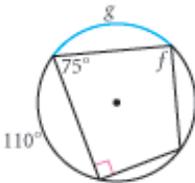
4. $h = ?$ (h)



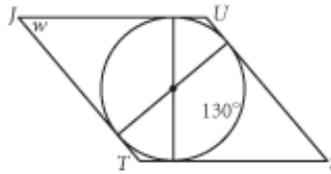
5. $d = ?$
 $e = ?$



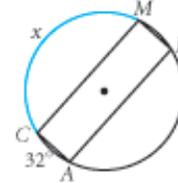
6. $f = ?$
 $g = ?$



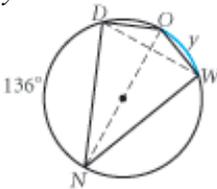
7. *JUST* is a rhombus.
 $w = ?$



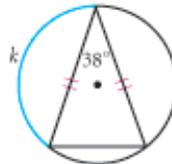
8. *CALM* is a rectangle.
 $x = ?$



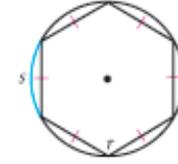
9. *DOWN* is a kite.
 $y = ?$



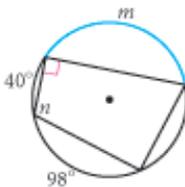
10. $k = ?$



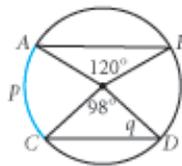
11. $r = ?$
 $s = ?$



12. $m = ?$
 $n = ?$



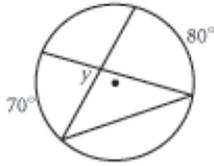
13. $\overline{AB} \parallel \overline{CD}$
 $p = ?$
 $q = ?$



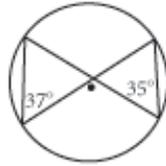
14. What is the sum of a , b , c , d , and e ? (h)



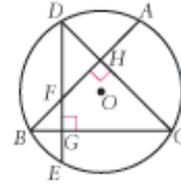
15. $y = ?$ 



16. **Developing Proof** What's wrong with this picture?



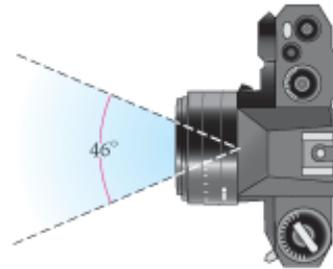
17. **Developing Proof** Explain why $\overline{AC} \cong \overline{CE}$.



18. How can you find the center of a circle, using only the corner of a piece of paper?

19. **Technology** Chris Chisholm, a high school student in Whitmore, California, used the Angles Inscribed in a Semicircle Conjecture to discover a simpler way to find the orthocenter in a triangle. Chris constructs a circle using one of the sides of the triangle as the diameter, then immediately finds an altitude to each of the triangle's other two sides. Use geometry software and Chris's method to find the orthocenter of a triangle. Does this method work on all kinds of triangles? 

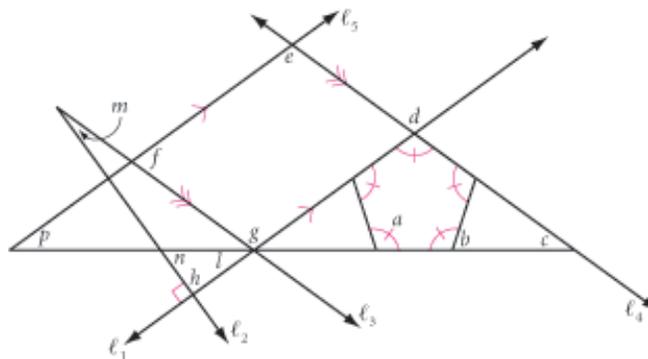
20. **Application** The width of a view that can be captured in a photo depends on the camera's *picture angle*. Suppose a photographer takes a photo of your class standing in one straight row with a camera that has a 46° picture angle. Draw a line segment to represent the row. Draw a 46° angle on a piece of patty paper. Locate at least eight different points on your paper where a camera could be positioned to include all the students, filling as much of the picture as possible. What is the locus of all such camera positions? What conjecture does this activity illustrate? 



21. **Technology** Construct a circle and a diameter. Construct a point on one of the semicircles, and construct two chords from it to the endpoints of the diameter to create a right triangle. Locate the midpoint of each of the two chords. Predict, then sketch the locus of the two midpoints as the vertex of the right angle is moved around the circle. Finally, use your computer to animate the point on the circle and trace the locus of the two midpoints. What do you get?

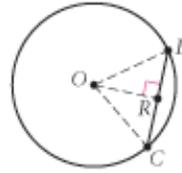
Review

22. Find the measure of each lettered angle.



23. Circle U passes through points $(3, 11)$, $(11, -1)$, and $(-14, 4)$. Find the coordinates of its center. Explain your method.

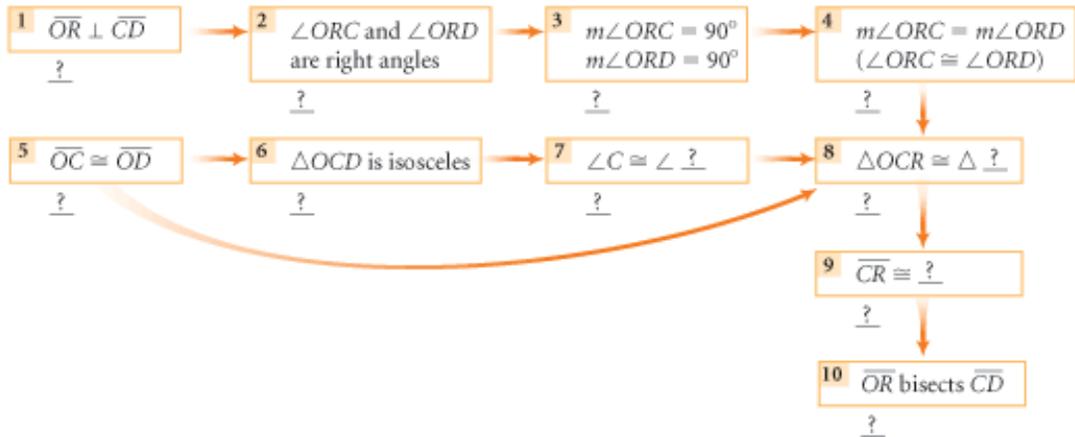
24. **Developing Proof** Complete the flowchart proof or write a paragraph proof of the Perpendicular to a Chord Conjecture: The perpendicular from the center of a circle to a chord is the bisector of the chord.



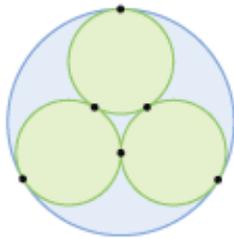
Given: Circle O with chord \overline{CD} , radii \overline{OC} and \overline{OD} , and $\overline{OR} \perp \overline{CD}$

Show: \overline{OR} bisects \overline{CD}

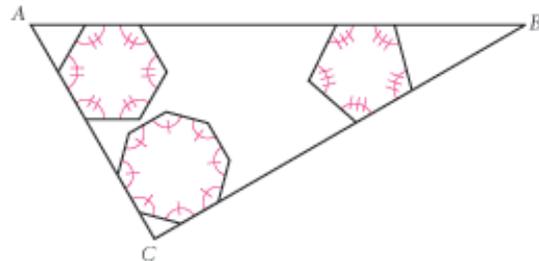
Flowchart Proof



25. **Construction** Use your construction tools to re-create this design of three congruent circles, all tangent to each other and internally tangent to a larger circle.



26. **Developing Proof** What's wrong with this picture?



IMPROVING YOUR REASONING SKILLS

Think Dinosaur

If the letter in the word *dinosaur* that is three letters after the word's second vowel is also found before the sixteenth letter of the alphabet, then print the word *dinosaur* horizontally. Otherwise, print the word *dinosaur* vertically and cross out the second letter after the first vowel.



Proving Circle Conjectures

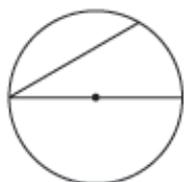
In the previous lesson you first discovered the Inscribed Angle Conjecture: The measure of an angle inscribed in a circle equals half the measure of its intercepted arc. You then discovered four other conjectures related to inscribed angles. In this lesson you will prove that all four of these conjectures are logical consequences of the Inscribed Angle Conjecture.

Mistakes are a fact of life.

It's the response to the error that counts.

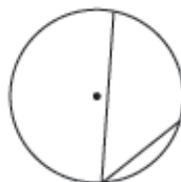
NIKKI GIOVANNI

First we must prove the Inscribed Angle Conjecture itself, but how? Let's use our reasoning strategies to make a plan. By thinking backward, we see that a central angle gives us something to compare an inscribed angle with. If one side of the inscribed angle is a diameter, then we can form a central angle by adding an auxiliary line. But what if the circle's center is not on the inscribed angle? There are three possible cases.



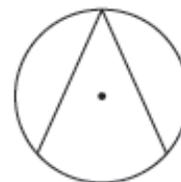
Case 1

The circle's center is on the angle.



Case 2

The center is outside the angle.



Case 3

The center is inside the angle.

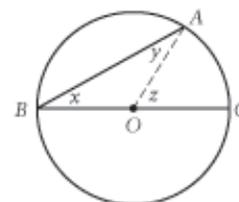
Let's break the problem into parts and consider one case at a time. We'll start with the easiest case first.

Case 1: The circle's center is on the inscribed angle.

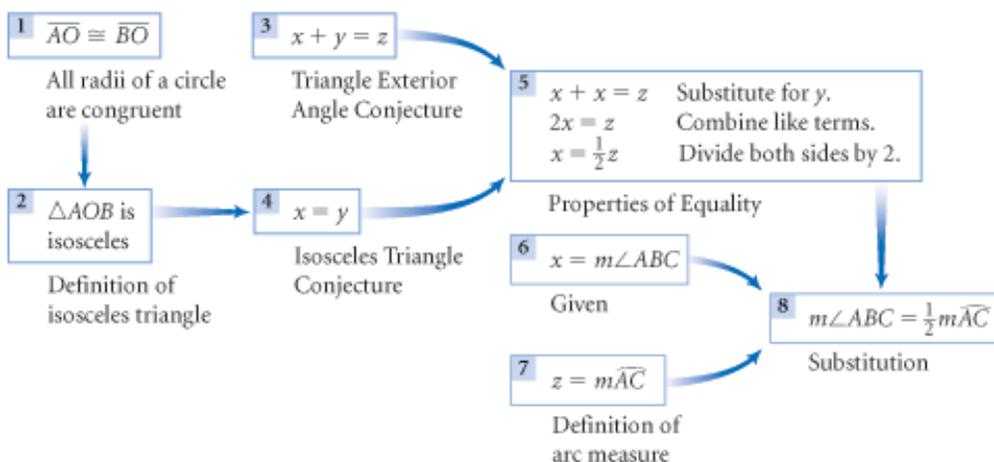
This proof uses the variables x , y , and z to represent the measures of the angles as shown in the diagram at right.

Given: Circle O with inscribed angle ABC on diameter \overline{BC}

Show: $m\angle ABC = \frac{1}{2}m\widehat{AC}$



Flowchart Proof of Case 1



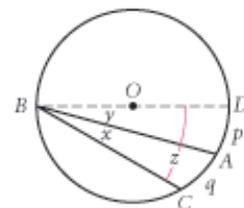


Developing Proof As a group, go through the flowchart proof of Case 1, one box at a time. What does each statement mean? How does it relate to the given diagram? How does the reason below the box support the statement? How do the arrows connect the flow of ideas? Discuss until all members of your group understand the logic of the proof. ■

The proof of Case 1 allows us now to prove the other two cases. By adding an auxiliary line, we can use the proof of Case 1 to show that the measures of the inscribed angles that *do* contain the diameter are half those of their intercepted arcs. The proof of Case 2 also requires us to accept angle addition and arc addition, or that the measures of adjacent angles and arcs on the same circle can be added.

Case 2: The circle's center is outside the inscribed angle.

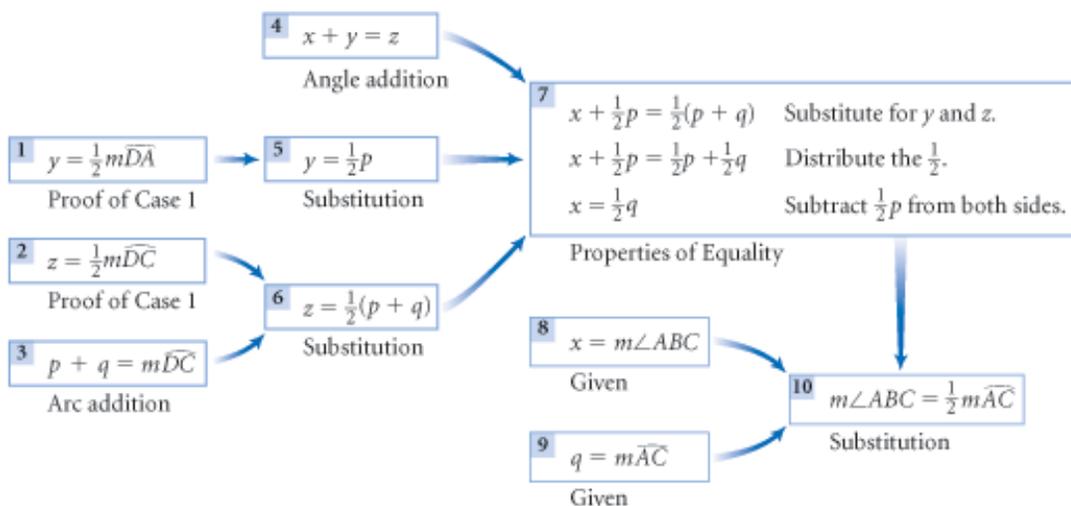
This proof uses x , y , and z to represent the measures of the angles, and p and q to represent the measures of the arcs, as shown in the diagram at right.



Given: Circle O with inscribed angle ABC on one side of diameter \overline{BD}

Show: $m\angle ABC = \frac{1}{2}m\widehat{AC}$

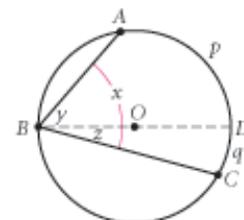
Flowchart Proof of Case 2



Developing Proof As a group, go through the flowchart proof of Case 2, as you did with Case 1, until all members of your group understand the logic of the proof. Then work together to create a flowchart proof for Case 3, similar to the proof of Case 2. ■

Case 3: The circle's center is inside the inscribed angle.

This proof uses x , y , and z to represent the measures of the angles, and p and q to represent the measures of the arcs, as shown in the diagram at right.



Given: Circle O with inscribed angle ABC whose sides lie on either side of diameter \overline{BD}

Show: $m\angle ABC = \frac{1}{2}m\widehat{AC}$

Having proved all three cases, we have now proved the Inscribed Angle Conjecture. You can now accept it as true to write proofs of other conjectures in the exercises.



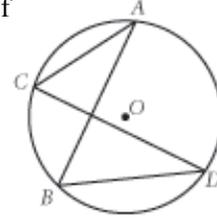
EXERCISES

Developing Proof In Exercises 1–4, the four conjectures are consequences of the Inscribed Angle Conjecture. Prove each conjecture by writing a paragraph proof or a flowchart proof. Use reasoning strategies, such as think backwards, apply previous conjectures and definitions, and break a problem into parts to develop your proofs.

1. Inscribed angles that intercept the same arc are congruent.

Given: Circle O with $\angle ACD$ and $\angle ABD$ inscribed in \overline{ACD}

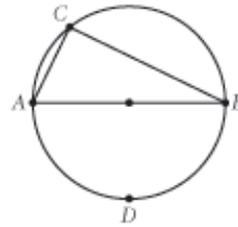
Show: $\angle ACD \cong \angle ABD$



2. Angles inscribed in a semicircle are right angles.

Given: Circle O with diameter \overline{AB} and $\angle ACB$ inscribed in semicircle ACB

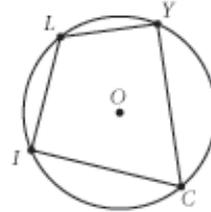
Show: $\angle ACB$ is a right angle



3. The opposite angles of a cyclic quadrilateral are supplementary.

Given: Circle O with inscribed quadrilateral $LICY$

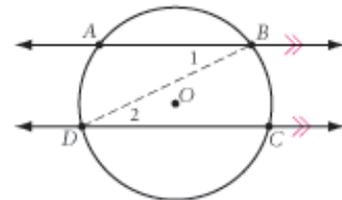
Show: $\angle L$ and $\angle C$ are supplementary



4. Parallel lines intercept congruent arcs on a circle.

Given: Circle O with chord \overline{BD} and $\overline{AB} \parallel \overline{CD}$

Show: $\overline{BC} \cong \overline{DA}$

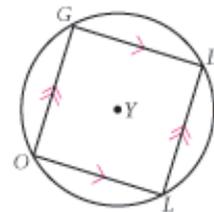


Developing Proof For Exercises 5–7, determine whether each conjecture is true or false. If the conjecture is false, draw a counterexample. If the conjecture is true, prove it by writing either a paragraph or flowchart proof.

5. If a parallelogram is inscribed within a circle, then the parallelogram is a rectangle.

Given: Circle Y with inscribed parallelogram $GOLD$

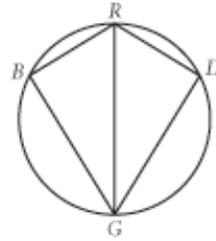
Show: $GOLD$ is a rectangle



6. If a kite is inscribed in a circle, then one of the diagonals of the kite is a diameter of the circle.

Given: $BRDG$ is a kite inscribed in a circle with $BR = RD$, $BG = DG$.

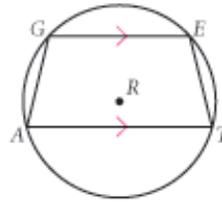
Show: \overline{RG} is a diameter.



7. If a trapezoid is inscribed within a circle, then the trapezoid is isosceles. \textcircled{h}

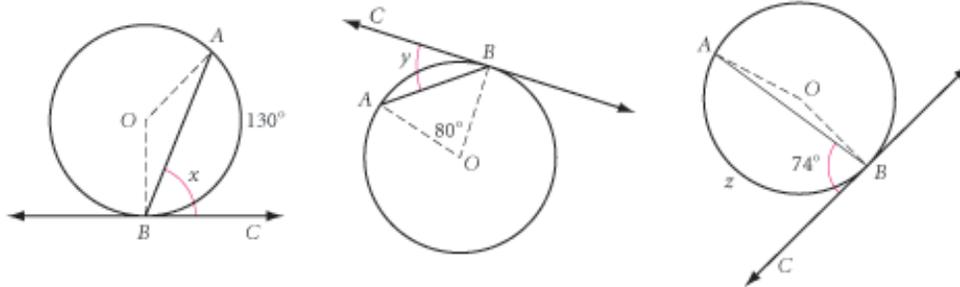
Given: Circle R with inscribed trapezoid $GATE$

Show: $GATE$ is an isosceles trapezoid



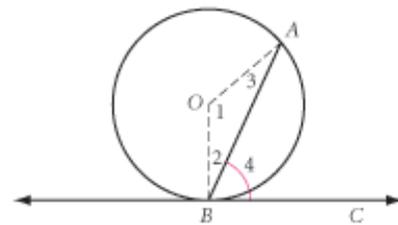
Review

8. **Mini-Investigation** Use what you know about isosceles triangles and the angle formed by a tangent and a radius to find the missing arc measure or angle measure in each diagram. Examine these cases to find a relationship between the measure of the angle formed by a tangent and chord at the point of tangency, $\angle ABC$, and the measure of the intercepted arc, \overline{AB} . Then copy and complete the conjecture below.



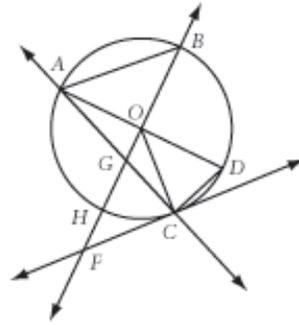
Conjecture: The measure of the angle formed by the intersection of a tangent and chord at the point of tangency is $\frac{1}{2}$. (Tangent-Chord Conjecture)

9. **Developing Proof** Given circle O with chord \overline{AB} and tangent \overline{BC} in the diagram at right, prove the conjecture you made in the last exercise. \textcircled{h}
10. For each of the statements below, choose the letter for the word that best fits (A stands for always, S for sometimes, and N for never). If the answer is S, give two examples, one showing how the statement can be true and one showing how the statement can be false.
- An equilateral polygon is (A/S/N) equiangular.
 - If a triangle is a right triangle, then the acute angles are (A/S/N) complementary.
 - The diagonals of a kite are (A/S/N) perpendicular bisectors of each other.
 - A regular polygon (A/S/N) has both reflectional symmetry and rotational symmetry.
 - If a polygon has rotational symmetry, then it (A/S/N) has more than one line of reflectional symmetry.

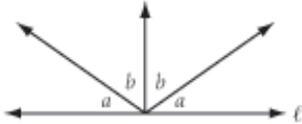


Match each term in Exercises 11–19 with one of the figures A–N.

- | | | |
|------------------------|--------------------|--------------------|
| 11. Minor arc | A. \overline{OC} | B. \overline{AB} |
| 12. Major arc | C. \overline{OF} | D. $\angle COD$ |
| 13. Semicircle | E. $\angle DAC$ | F. $\angle ACF$ |
| 14. Central angle | G. \overline{CF} | H. \overline{AC} |
| 15. Inscribed angle | I. \overline{GB} | J. \widehat{BAD} |
| 16. Chord | K. \widehat{ABD} | L. \overline{CD} |
| 17. Secant | M. $\triangle OCD$ | N. $\triangle ACD$ |
| 18. Tangent | | |
| 19. Inscribed triangle | | |



20. **Developing Proof** Explain why a and b are complementary.



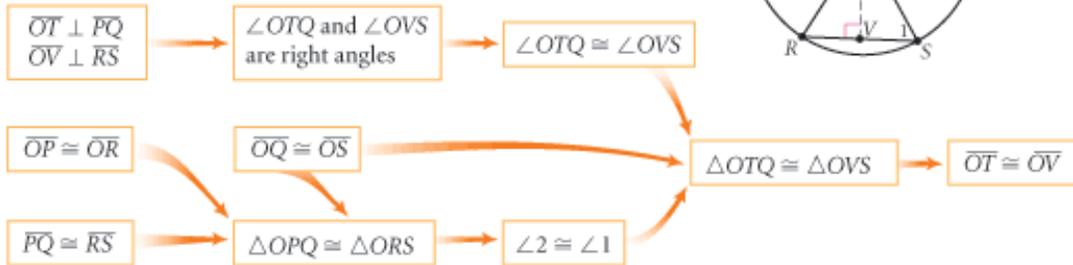
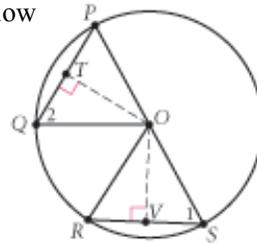
21. What is the probability of randomly selecting three collinear points from the points in the 3-by-3 grid below? \textcircled{h}



22. **Developing Proof** Use the diagram at right and the flowchart below to write a paragraph proof explaining why two congruent chords in a circle are equidistant from the center of the circle. \textcircled{h}

Given: Circle O with $\overline{PQ} \cong \overline{RS}$ and $\overline{OT} \perp \overline{PQ}$ and $\overline{OV} \perp \overline{RS}$

Show: $\overline{OT} \cong \overline{OV}$



IMPROVING YOUR VISUAL THINKING SKILLS

Rolling Quarters

One of two quarters remains motionless while the other rotates around it, never slipping and always tangent to it. When the rotating quarter has completed a turn around the stationary quarter, how many turns has it made around its own center point? Try it!



The Circumference/ Diameter Ratio

No human investigations can ever be called true science without going through mathematical tests.

LEONARDO DA VINCI

The distance around a polygon is called the perimeter. The distance around a circle is called the **circumference**. Here is a nice visual puzzle. Which is greater, the height of a tennis-ball can or the circumference of the can? The height is approximately three tennis-ball diameters tall. The diameter of the can is approximately one tennis-ball diameter. If you have a tennis-ball can handy, try it. Wrap a string around the can to measure its circumference, then compare this measurement with the height of the can. Surprised?



If you actually compared the measurements, you discovered that the circumference of the can is greater than three diameters of the can. In this lesson you are going to discover (or perhaps rediscover) the relationship between the diameter and the circumference of every circle. Once you know this relationship, you can measure a circle's diameter and calculate its circumference.

If you measure the circumference and diameter of a circle and divide the circumference by the diameter, you get a number slightly larger than 3. The more accurate your measurements, the closer your ratio will come to a special number called π (pi), pronounced "pie," like the dessert.

History

CONNECTION

In 1897, the Indiana state assembly tried to legislate the value of π . The vague language of the state's House Bill No. 246, which became known as the "Indiana Pi Bill," implies several different incorrect values for π —3.2, 3.232, 3.236, 3.24, and 4. With a unanimous vote of 67-0, the House passed the bill to the state senate, where it was postponed indefinitely.



Investigation A Taste of Pi

You will need

- several round objects (cans, mugs, bike wheel, plates)
- a meterstick or metric measuring tape
- sewing thread or thin string

In this investigation you will find an approximate value of π by measuring circular objects and calculating the ratio of the circumference to the diameter. Let's see how close you come to the actual value of π .



- Step 1 | Measure the circumference of each round object by wrapping the measuring tape, or string, around its perimeter. Then measure the diameter of each object with the meterstick or tape. Record each measurement to the nearest millimeter (tenth of a centimeter).
- Step 2 | Make a table like the one below and record the circumference (C) and diameter (d) measurements for each round object.

Object	Circumference (C)	Diameter (d)	Ratio $\frac{C}{d}$
Can			
Mug			
Wheel			

- Step 3 | Calculate the ratio $\frac{C}{d}$ for each object. Record the answers in your table.
- Step 4 | Calculate the average of your ratios of $\frac{C}{d}$.

Compare your average with the averages of other groups. Are the $\frac{C}{d}$ ratios close? You should now be convinced that the ratio $\frac{C}{d}$ is very close to 3 for every circle. We define π as the ratio $\frac{C}{d}$. If you solve this formula for C , you get a formula for the circumference of a circle in terms of the diameter, d . The diameter is twice the radius ($d = 2r$), so you can *also* get a formula for the circumference in terms of the radius, r .

- Step 5 | Copy and complete the conjecture.

Circumference Conjecture

C-65

If C is the circumference and d is the diameter of a circle, then there is a number such that $C = \underline{\quad ? \quad}$. If $d = 2r$ where r is the radius, then $C = \underline{\quad ? \quad}$.

Mathematics

CONNECTION

The number π is an irrational number—its decimal form never ends and the pattern of digits does not repeat. The symbol π is a letter of the Greek alphabet. Perhaps no other number has more fascinated mathematicians throughout history. Mathematicians in ancient Egypt used $(\frac{4}{3})^2$ as their approximation of circumference to diameter. Early Chinese and Hindu mathematicians used $\sqrt{10}$. By 408 C.E., Chinese mathematicians were using $\frac{355}{113}$. Today, computers have calculated approximations of π to billions of decimal places, and there are websites devoted to π ! See www.keymath.com/DG

Accurate approximations of π have been of more interest intellectually than practically. Still, what would a carpenter say if you asked her to cut a board 3π feet long? Most calculators have a π button that gives π to eight or ten decimal places. You can use this value for most calculations, then round your answer to a specified decimal place. If your calculator doesn't have a π button, or if you don't have access to a calculator, use the value 3.14 for π . If you're asked for an exact answer instead of an approximation, state your answer in terms of π .

How do you use the Circumference Conjecture? Let's look at two examples.

EXAMPLE A | If a circle has diameter 3.0 meters, what is the circumference? Use a calculator and state your answer to the nearest 0.1 meter.

► **Solution**

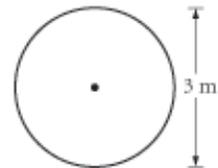
$$C = \pi d$$

Original formula.

$$C = \pi(3.0)$$

Substitute the value of d .

In terms of π , the answer is 3π . The circumference is about 9.4 meters.



EXAMPLE B | If a circle has circumference 12π meters, what is the radius?

► **Solution**

$$C = 2\pi r$$

Original formula.

$$12\pi = 2\pi r$$

Substitute the value of C .

$$r = 6$$

Solve.

The radius is 6 meters.



EXERCISES

You will need



A calculator
for Exercises 7–10

Use the Circumference Conjecture to solve Exercises 1–12. In Exercises 1–6, leave your answer in terms of π .

1. If $C = 5\pi$ cm, find d .
2. If $r = 5$ cm, find C .
3. If $C = 24$ m, find r .
4. If $d = 5.5$ m, find C .
5. If a circle has a diameter of 12 cm, what is its circumference?
6. If a circle has a circumference of 46π m, what is its diameter?

In Exercises 7–10, use a calculator. Round your answer to the nearest 0.1 unit. Use the symbol \approx to show that your answer is an approximation.

7. If $d = 5$ cm, find C .
8. If $r = 4$ cm, find C .
9. If $C = 44$ m, find r . 
10. What's the circumference of a bicycle wheel with a 27-inch diameter?



11. If the distance from the center of a Ferris wheel to one of the seats is approximately 90 feet, what is the distance traveled by a seated person, to the nearest foot, in one revolution?
12. If a circle is inscribed in a square with a perimeter of 24 cm, what is the circumference of the circle? 
13. If a circle with a circumference of 16π inches is circumscribed about a square, what is the length of a diagonal of the square?
14. Each year a growing tree adds a new ring to its cross section. Some years the ring is thicker than others. Why do you suppose this happens?



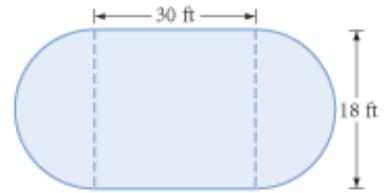
Suppose the average thickness of growth rings in the Flintstones National Forest is 0.5 cm. About how old is “Old Fred,” a famous tree in the forest, if its circumference measures 766 cm?

Science CONNECTION

Trees can live hundreds to thousands of years, and we can determine the age of one tree by counting its growth rings. A pair of rings—a light ring formed in the spring and summer and a dark one formed in the fall and early winter—represent the growth for one year. We can learn a lot about the climate of a region over a period of years by studying tree growth rings. This study is called *dendroclimatology*.

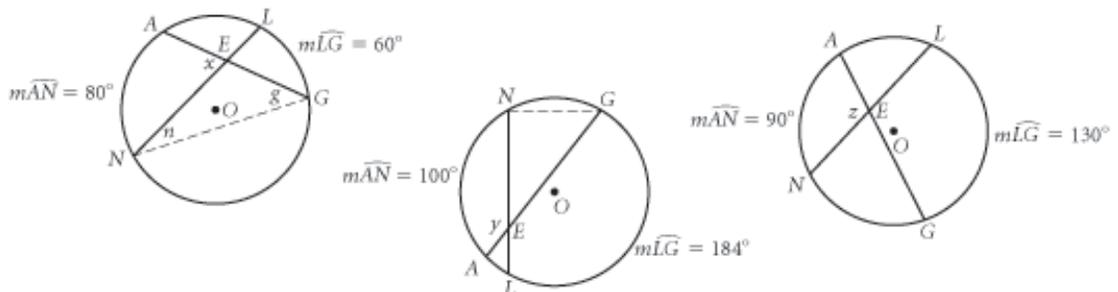


15. Pool contractor Peter Tileson needs to determine the number of 1-inch tiles to put around the edge of a pool. The pool is a rectangle with two semicircular ends as shown. How many tiles will he need?



Review

16. **Mini-Investigation** Use what you know about inscribed angles and exterior angles of a triangle to find the missing angle measures in each diagram. Examine these cases to find a relationship between the measure of $\angle AEN$ and the measures of the two intercepted arcs, \widehat{AN} and \widehat{LG} . Then copy and complete the conjecture below.



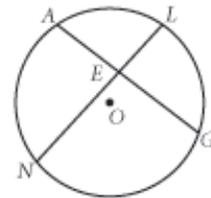
Conjecture: The measure of an angle formed by two intersecting chords is $\frac{1}{2}$ the sum of the measures of the two intercepted arcs.
(Intersecting Chords Conjecture)

- ▶ For an interactive version of this mini-investigation, see the **Dynamic Geometry Exploration** Intersecting Chords at www.Keymath.com/DG.



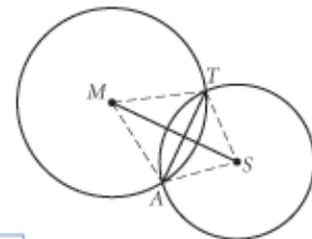
[keymath.com/DG](http://www.Keymath.com/DG)

17. **Developing Proof** Given circle O with chords \overline{AG} and \overline{LN} in the diagram at right, prove the conjecture you made in the last exercise. Start by drawing the auxiliary line \overline{NG} .
18. **Developing Proof** Prove the conjecture: If two circles intersect at two points, then the segment connecting the centers is the perpendicular bisector of the common chord, the segment connecting the points of intersection.



Given: Circle M and circle S intersect at points A and T with radii $\overline{MA} \cong \overline{MT}$ and $\overline{SA} \cong \overline{ST}$

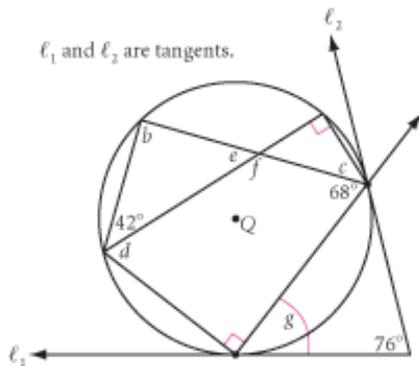
Show: \overline{MS} is the perpendicular bisector of \overline{AT}



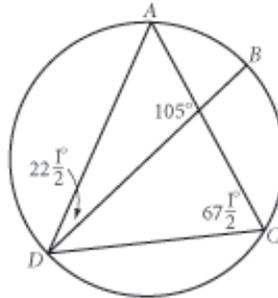
Flowchart Proof



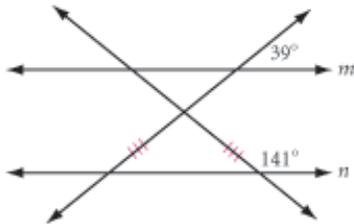
19. Trace the figure below. Calculate the measure of each lettered angle.



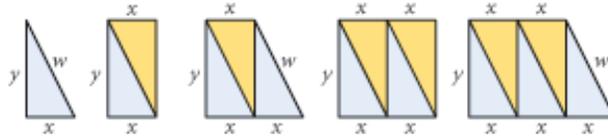
20. What is the probability that a flea strolling along the circle shown below will stop randomly on either \overline{AB} or \overline{CD} ? (Because you're probably not an expert in flea behavior, assume the flea will stop exactly once.)



21. **Developing Proof** Explain why m is parallel to n . (H)



22. Assume the pattern below will continue. Write an expression for the perimeter of the tenth shape in this picture pattern.



project

NEEDLE TOSS

If you randomly toss a needle on lined paper, what is the probability that the needle will land on one or more lines? What is the probability it will not land on any lines? The length of the needle and the distance between the lines will affect these probabilities.

Start with a toothpick of any length L as your “needle” and construct parallel lines a distance L apart. Write your predictions, then experiment. Using N as the number of times you dropped the needle, and C as the number of times the needle crossed the line, enter your results into the expression $2N/C$. As you drop the needle more and more times, the value of this expression seems to be getting close to what number?

Your project should include

- ▶ Your predictions and data.
- ▶ The calculated probabilities and your prediction of the theoretical probability.
- ▶ Any other interesting observations or conclusions.

Fathom 2
Dynamic Data Software

Using Fathom, you can simulate many experiments. See the demonstration Buffon's Needle that comes with the software package.

Around the World

Many application problems are related to π . Satellite orbits, the wheels of a vehicle, tree trunks, and round pizzas are just a few of the real-world examples that involve the circumference of circles. Here is a famous example from literature.

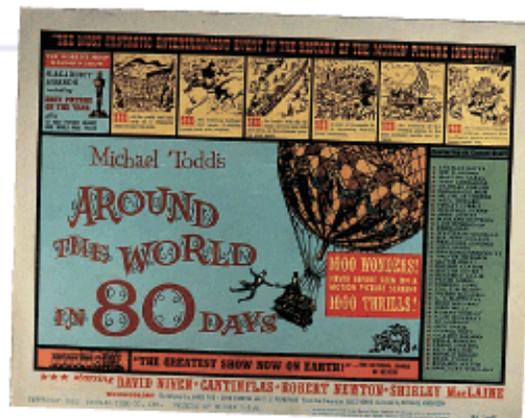
Love is like π —natural, irrational, and very important.

LISA HOFFMAN

Literature CONNECTION

In the novel *Around the World in Eighty Days* (1873), Jules Verne (1828–1905) recounts the adventures of brave Phileas Fogg and his servant Passerpartout.

They begin their journey when Phileas bets his friends that he can make a trip around the world in 80 days. Phileas's precise behavior, such as monitoring the temperature of his shaving water or calculating the exact time and location of his points of travel, reflects Verne's interest in the technology boom of the late nineteenth century. His studies in geology, engineering, and astronomy aid the imaginative themes in this and his other novels, including *A Journey to the Center of the Earth* (1864) and *Twenty Thousand Leagues Under the Sea* (1870).



EXAMPLE

If the diameter of Earth is 8000 miles, find the average speed in miles per hour Phileas Fogg needs to circumnavigate Earth about the equator in 80 days.

► Solution

To find the speed, you need to know the distance and the time. The distance around the equator is equal to the circumference C of a circle with a diameter of 8,000 miles.

$$\begin{aligned} C &= \pi d \\ &= \pi(8,000) \\ &\approx 25,133 \end{aligned}$$

The equation for circumference.

Substitute 8,000 for d .

Round to nearest mile.

So, Phileas must travel 25,133 miles in 80 days. To find the speed v in mi/h, you need to divide distance by time and convert days into hours.

$$\begin{aligned} v &= \frac{\text{distance}}{\text{time}} \\ &\approx \frac{25,133 \text{ mi}}{80 \text{ days}} \cdot \frac{1 \text{ day}}{24 \text{ h}} \\ &\approx 13 \text{ mi/h} \end{aligned}$$

The formula for speed, or velocity.

Substitute values and convert units of time.

Evaluate and round to the nearest mile per hour.

If Earth's diameter were *exactly* 8,000 miles, you could evaluate $\frac{8,000\pi}{80 \cdot 24}$ and get an exact answer of $\frac{25\pi}{6}$ in terms of π .

EXERCISES

You will need



In Exercises 1–6, round answers to the nearest unit. You may use 3.14 as an approximate value of π . If you have a π button on your calculator, use that value and then round your final answer.

1. A satellite in a nearly circular orbit is 2000 km above Earth's surface. The radius of Earth is approximately 6400 km. If the satellite completes its orbit in 12 hours, calculate the speed of the satellite in kilometers per hour. h
2. Wilbur Wrong is flying his remote-control plane in a circle with a radius of 28 meters. His brother, Orville Wrong, clocks the plane at 16 seconds per revolution. What is the speed of the plane? Express your answer in meters per second. The brothers may be wrong, but you could be right!
3. Here is a tiring problem. The diameter of a car tire is approximately 60 cm (0.6 m). The warranty is good for 70,000 km. About how many revolutions will the tire make before the warranty is up? More than a million? A billion? (1 km = 1000 m)
4. If the front tire of this motorcycle has a diameter of 50 cm (0.5 m), how many revolutions will it make if it is pushed 1 km to the nearest gas station? In other words, how many circumferences of the circle are there in 1000 meters?
5. Goldi's Pizza Palace is known throughout the city. The small Baby Bear pizza has a 6-inch radius and sells for \$9.75. The savory medium Mama Bear pizza sells for \$12.00 and has an 8-inch radius. The large Papa Bear pizza is a hefty 20 inches in diameter and sells for \$16.50. The edge is stuffed with cheese, and it's the best part of a Goldi's pizza. What size has the most pizza edge per dollar? What is the circumference of this pizza?



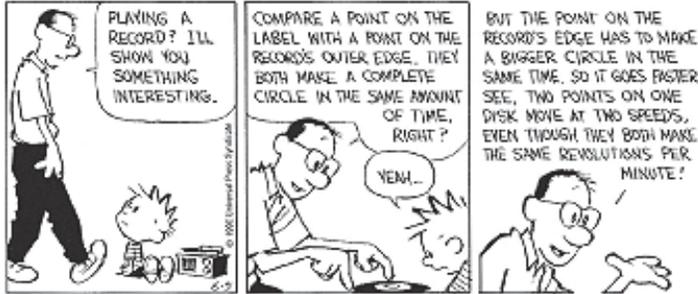
6. Felicia is a park ranger, and she gives school tours through the redwoods in a national park. Someone in every tour asks, "What is the diameter of the giant redwood tree near the park entrance?" Felicia knows that the arm span of each student is roughly the same as his or her height. So in response, Felicia asks a few students to arrange themselves around the circular base of the tree so that by hugging the tree with arms outstretched, they can just touch fingertips to fingertips. She then asks the group to calculate the diameter of the tree.



In one group, four students with heights of 138 cm, 136 cm, 128 cm, and 126 cm were able to ring the tree. What is the approximate diameter of the redwood?

7. **Application** Zach wants a circular table so that 12 chairs, each 16 inches wide, can be placed around it with at least 8 inches between chairs. What should be the diameter of the table? Will the table fit in a 12-by-14-foot dining room? Explain.

Calvin and Hobbes



by Bill Watterson



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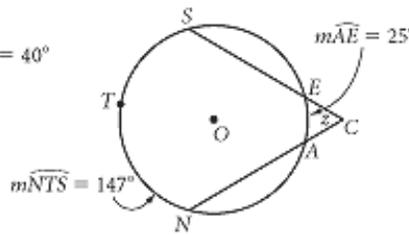
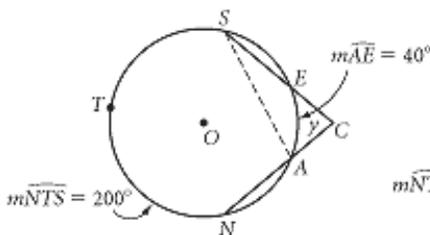
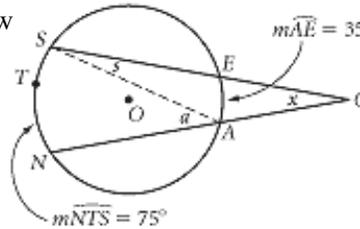
8. A 45 rpm record has a 7-inch diameter and spins at 45 revolutions per minute. A 33 rpm record has a 12-inch diameter and spins at 33 revolutions per minute. Find the difference in speeds of a point on the edge of a 33 rpm record to that of a point on the edge of a 45 rpm record, in ft/s.

Recreation CONNECTION

Using tinfoil records, Thomas Edison (1847–1931) invented the phonograph, the first machine to play back recorded sound, in 1877. Commonly called record players, they weren't widely reproduced until high-fidelity amplification (hi-fi) and advanced speaker systems came along in the 1930s. In 1948, records could be played at slower speeds to allow more material on the disc, creating longer-playing records (LPs). When compact discs became popular in the early 1990s, most record companies stopped making LPs. Some disc jockeys still use records instead of CDs.

Review

9. **Mini-Investigation** Use what you know about inscribed angles and exterior angles of a triangle to find the missing angle measures in each diagram. Examine these cases to find a relationship between the measure of the angle formed by two intersecting secants, $\angle ECA$, and the measures of the two intercepted arcs, \widehat{NTS} and \widehat{AE} . Then copy and complete the conjecture below.



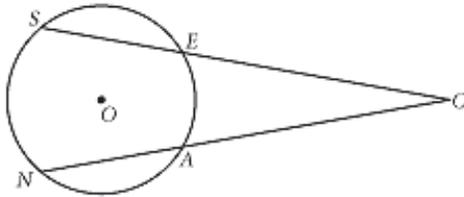
Conjecture: The measure of an angle formed by two secants that intersect outside a circle is $\frac{?}{2}$. (Intersecting Secants Conjecture)

▶ For an interactive version of this mini-investigation, see the **Dynamic Geometry Exploration** Intersecting Secants at www.keymath.com/DG.



keymath.com/DG

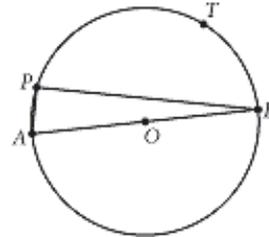
10. **Developing Proof** Given circle O with secants \overleftrightarrow{SE} and \overleftrightarrow{NA} in the diagram below, prove the conjecture you made in the last exercise. Start by drawing auxiliary line \overleftrightarrow{SA} . 



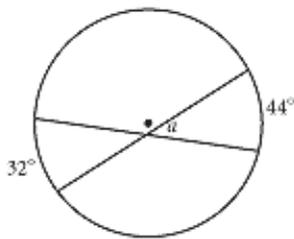
11. **Developing Proof** Explain why a equals b .



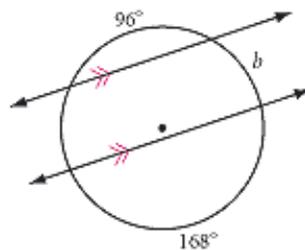
12. As P moves from A to B along the semicircle \overline{ATB} , which of these measures constantly increases?
- The perimeter of $\triangle ABP$
 - The distance from P to \overline{AB}
 - $m\angle ABP$
 - $m\angle APB$



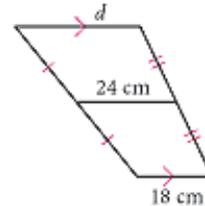
13. $a = ?$



14. $b = ?$



15. $d = ?$



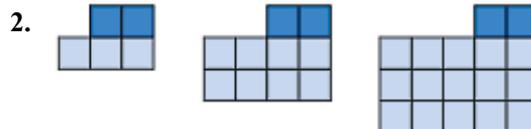
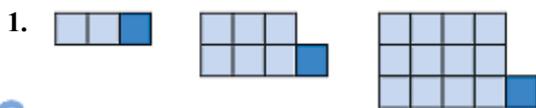
16. A helicopter has three blades each measuring about 26 feet. What is the speed in feet per second at the tips of the blades when they are moving at 400 rpm?
17. Two sides of a triangle are 24 cm and 36 cm. Write an inequality that represents the range of values for the third side. Explain your reasoning.



IMPROVING YOUR VISUAL THINKING SKILLS

Picture Patterns I

Draw the next picture in each pattern. Then write the rule for the total number of squares in the n th picture of the pattern.



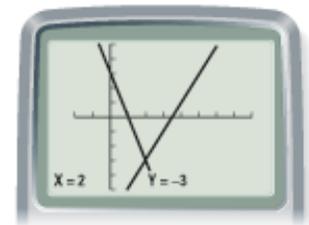
Solving Systems of Linear Equations

A system of equations is a set of two or more equations with the same variables. The solution of a system is the set of values that makes all the equations in the system true. For example, the system of equations below has solution $(2, -3)$. Verify this by substituting 2 for x and -3 for y in both equations.

$$\begin{cases} y = 2x - 7 \\ y = -3x + 3 \end{cases}$$

Graphically, the solution of a system is the point of intersection of the graphs of the equations.

You can estimate the solution of a system by graphing the equations. However, the point of intersection may not have convenient integer coordinates. To find the exact solution, you can use algebra. Examples A and B review how to use the *substitution* and *elimination* methods for solving systems of equations.



EXAMPLE A | Solve the system $\begin{cases} 3y = 12x - 21 \\ 12x + 2y = 1 \end{cases}$

► **Solution**

You could use either method to solve this system of equations, but substitution may be the most direct. If you solve the first equation for y , then you can substitute the resulting expression for y into the second equation. Start by solving the first equation for y to get $y = 4x - 7$. Now, substitute the expression $4x - 7$ from the resulting equation for y in the second original equation.

$$12x + 2y = 1 \quad \text{Second original equation.}$$

$$12x + 2(4x - 7) = 1 \quad \text{Substitute } 4x - 7 \text{ for } y.$$

$$x = \frac{3}{4} \quad \text{Solve for } x.$$

To find y , substitute $\frac{3}{4}$ for x in either original equation.

$$3y = 12\left(\frac{3}{4}\right) - 21 \quad \text{Substitute } \frac{3}{4} \text{ for } x \text{ in the first original equation.}$$

$$y = -4 \quad \text{Solve for } y.$$

The solution of the system is $\left(\frac{3}{4}, -4\right)$. Verify by substituting these values for x and y in each of the original equations.

EXAMPLE B | Solve the system $\begin{cases} 140a + 60b = 40 \\ 200a + 30b = 85 \end{cases}$

► **Solution**

The way these equations are arranged, the variables line up. So elimination is probably the most direct method for solving this system. More importantly, it would be difficult to solve either equation for either variable. Using substitution would involve many fractions.

Solving a system by elimination involves adding or subtracting multiples of the equations to eliminate one of the variables. To solve this system, first multiply both sides of the second equation by 2 so that you have the same quantities of variable b .

$$\begin{cases} 140a + 60b = 40 \\ 200a + 30b = 85 \end{cases} \rightarrow \begin{cases} 140a + 60b = 40 \\ 400a + 60b = 170 \end{cases}$$

Now subtract the second equation from the first to eliminate b . Then solve for a .

$$\begin{array}{r} 140a + 60b = 40 \\ -(400a + 60b = 170) \\ \hline -260a = -130 \\ a = \frac{1}{2} \end{array}$$

To find the value of b , substitute $\frac{1}{2}$ for a in either original equation.

$$140\left(\frac{1}{2}\right) + 60b = 40 \quad \text{Substitute } \frac{1}{2} \text{ for } a \text{ in the first equation.}$$

$$b = -\frac{1}{2} \quad \text{Solve for } b.$$

The solution is $a = \frac{1}{2}$ and $b = -\frac{1}{2}$.

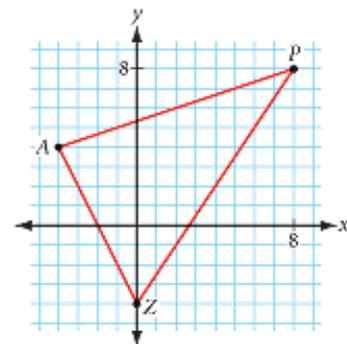
You have constructed points of concurrency, which are intersections of lines in a triangle. You can also find these on a coordinate plane. Suppose you know the coordinates of the vertices of a triangle. How can you find the coordinates of the circumcenter? You can graph the triangle, construct the perpendicular bisectors of the sides, and then *estimate* the coordinates of the point of concurrency. However, to find the exact coordinates, you need to use algebra. Let's look at an example.

EXAMPLE C

Find the coordinates of the circumcenter of $\triangle ZAP$ with $Z(0, -4)$, $A(-4, 4)$, and $P(8, 8)$.

► **Solution**

To find the coordinates of the circumcenter, you need to write equations for the perpendicular bisectors of two of the sides of the triangle and then find the point where the bisectors intersect.



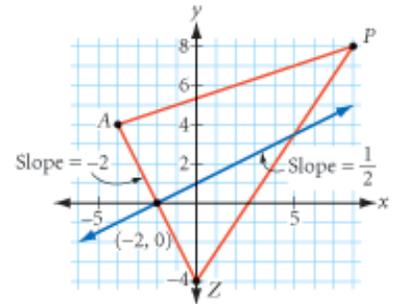
To review midpoint see pages 36–37. To review slope see pages 135–136.

To review slopes of perpendicular lines see pages 167–168. To review the slope-intercept form see pages 287–289.

To find the equation for the perpendicular bisector of \overline{ZA} , first find the midpoint of \overline{ZA} , then find its slope.

$$\begin{aligned} \text{Midpoint of } \overline{ZA} &= \left(\frac{0 + (-4)}{2}, \frac{-4 + 4}{2} \right) \\ &= (-2, 0) \end{aligned}$$

$$\text{Slope of } \overline{ZA} = \frac{4 - (-4)}{-4 - 0} = \frac{8}{-4} = -2$$



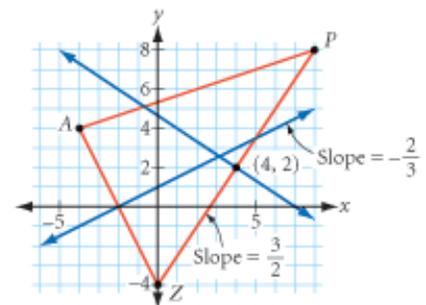
The slope of the perpendicular bisector of \overline{ZA} is the opposite reciprocal of -2 , or $\frac{1}{2}$, and it passes through point $(-2, 0)$. Substituting these values into the slope-intercept form of a line and solving for the y -intercept gives the equation of the perpendicular bisector.

$$y = mx + b$$

$$0 = \frac{1}{2}(-2) + b$$

$$1 = b$$

$$y = \frac{1}{2}x + 1$$

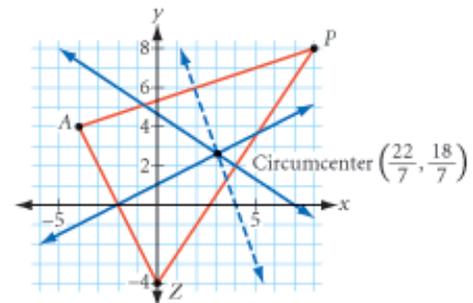


You can use the same technique to find the equation of the perpendicular bisector of \overline{ZP} . The midpoint of \overline{ZP} is $(4, 2)$, and the slope is $\frac{3}{2}$. So the slope of the perpendicular bisector of \overline{ZP} is $-\frac{2}{3}$ and it passes through the point $(4, 2)$. The equation of the perpendicular bisector is $y = -\frac{2}{3}x + \frac{14}{3}$.

Since all the perpendicular bisectors intersect at the same point, you can solve these two equations to find that point.

$$\begin{cases} y = \frac{1}{2}x + 1 & \text{Perpendicular bisector of } \overline{ZA} \\ y = -\frac{2}{3}x + \frac{14}{3} & \text{Perpendicular bisector of } \overline{ZP} \end{cases}$$

The circumcenter is $\left(\frac{22}{7}, \frac{18}{7}\right)$. The steps of the solution are left as an exercise. You can verify this result by finding the equation of the perpendicular bisector of \overline{AP} and making sure that these values satisfy it.





EXERCISES

Solve each system of equations algebraically.

1.
$$\begin{cases} y = -2x + 2 \\ 6x + 2y = 3 \end{cases}$$

2.
$$\begin{cases} -4x + 3y = 3 \\ 7x - 9y = 6 \end{cases}$$

3.
$$\begin{cases} 5x - y = -1 \\ 15x = 2y \end{cases}$$

4.
$$\begin{cases} x + 2y = 3 \\ 2x - y = 16 \end{cases}$$

5.
$$\begin{cases} x + 3y = 6 \\ \frac{1}{3}x + y = 2 \end{cases}$$

6.
$$\begin{cases} 2x + y = 9 \\ y = -2x - 1 \end{cases}$$

7. For Exercises 4, 5, and 6, graph the system of equations and explain how the solution relates to the graph.

8. Solve the system of equations in Example C to show that the answer given is correct. Explain why you need to use only two perpendicular bisectors, not three.

9. A snowboard rental company offers two different rental plans. Plan A offers \$4/h for the rental and a \$20 lift ticket. Plan B offers \$7/h for the rental and a free lift ticket.

- Write the two equations that represent the costs for the two plans, using x for the number of hours. Solve for x and y .
- Graph the two equations. What does the point of intersection represent?
- Which is the better plan if you intend to snowboard for 5 hours? What is the most number of hours of snowboarding you can get for \$50?



10. The lines $y = 3 + \frac{2}{3}x$, $y = -\frac{1}{3}x$, and $y = -\frac{4}{3}x + 3$ intersect to form a triangle. Find the vertices of the triangle.

11. The vertices of parallelogram $ABCD$ are $A(2, 3)$, $B(8, 4)$, $C(10, 9)$, and $D(4, 8)$.

- Write the system of linear equations created by the diagonals of $ABCD$.
- Solve the system to find the point of intersection of the diagonals.
- Calculate the midpoint of each diagonal.
- Explain how your results to parts b and c confirm the Parallelogram Diagonals Conjecture.

12. Triangle RES has vertices $R(0, 0)$, $E(4, -6)$, and $S(8, 4)$. Find the equation of the perpendicular bisector of \overline{RE} .

In Exercises 13–15, find the coordinates of the circumcenter of each triangle.

13. Triangle TRM with vertices $T(-2, 1)$, $R(4, 3)$, and $M(-4, -1)$

14. Triangle FGH with vertices $F(0, -6)$, $G(3, 6)$, and $H(12, 0)$

15. Right triangle MNO with vertices $M(-4, 0)$, $N(0, 5)$, and $O(10, -3)$

16. For a right triangle, there is a shorter method for finding the circumcenter. What is it? Explain. (*Hint*: Graphing your work may help you recognize the method.)

Arc Length

Some people transform the sun into a yellow spot, others transform a yellow spot into the sun.

PABLO PICASSO

You have learned that the *measure* of a minor arc is equal to the measure of its central angle. On a clock, the measure of the arc from 12:00 to 4:00 is equal to the measure of the angle formed by the hour and minute hands. A circular clock is divided into 12 equal arcs, so the measure of each hour is $\frac{360^\circ}{12}$, or 30° . The measure of the arc from 12:00 to 4:00 is four times 30° , or 120° .

Notice that because the minute hand is longer, the tip of the minute hand must travel farther than the tip of the hour hand even though they both move 120° from 12:00 to 4:00. So the arc *length* is different even though the arc *measure* is the same!

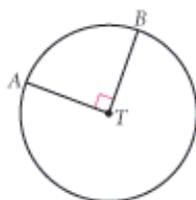


Let's take another look at the arc measure.

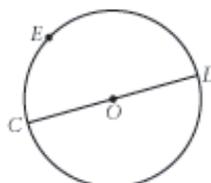
EXAMPLE A

What fraction of its circle is each arc?

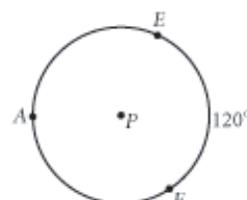
a. \widehat{AB} is what fraction of circle T ?



b. \widehat{CED} is what fraction of circle O ?



c. \widehat{EF} is what fraction of circle P ?



► Solution

In part a, you probably “just knew” that the arc is one-fourth of the circle because you have seen one-fourth of a circle so many times. Why is it one-fourth? The arc measure is 90° , a full circle measures 360° , and $\frac{90^\circ}{360^\circ} = \frac{1}{4}$. The arc in part b is half of the circle because $\frac{180^\circ}{360^\circ} = \frac{1}{2}$. In part c, you may or may not have recognized right away that the arc is one-third of the circle. The arc is one-third of the circle because $\frac{120^\circ}{360^\circ} = \frac{1}{3}$.

Cultural CONNECTION

This modern mosaic shows the plan of an ancient Mayan observatory. On certain days of the year light would shine through openings, indicating the seasons. This sculpture includes blocks of marble carved into arcs of concentric circles.



What do these fractions have to do with arc length? If you traveled halfway around a circle, you'd cover $\frac{1}{2}$ of its perimeter, or circumference. If you went a quarter of the way around, you'd travel $\frac{1}{4}$ of its circumference. The **arc length** is some fraction of the circumference of its circle.

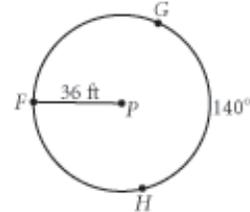
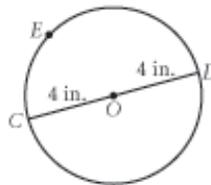
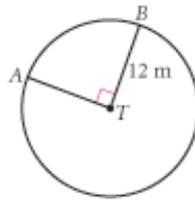
The measure of an arc is calculated in units of degrees, but arc length is calculated in units of distance.



Investigation Finding the Arcs

In this investigation you will find a method for calculating the arc length.

Step 1 For \overline{AB} , \overline{CED} , and \overline{GH} , find what fraction of the circle each arc is.



Step 2 Find the circumference of each circle.

Step 3 Combine the results of Steps 1 and 2 to find the length of each arc.

Step 4 Share your ideas for finding the length of an arc. Generalize this method for finding the length of *any* arc, and state it as a conjecture.

Arc Length Conjecture

C-66

The length of an arc equals the $\frac{?}{?}$.

How do you use this new conjecture? Let's look at a few examples.

EXAMPLE B

If the radius of the circle is 24 cm and $m\angle BTA = 60^\circ$, what is the length of \overline{AB} ?

► Solution

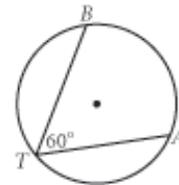
$m\angle BTA = 60^\circ$, so $m\overline{AB} = 120^\circ$ by the Inscribed Angle Conjecture. Then $\frac{120}{360} = \frac{1}{3}$, so the arc length is $\frac{1}{3}$ of the circumference, by the Arc Length Conjecture.

$$\begin{aligned} \text{arc length} &= \frac{1}{3}C \\ &= \frac{1}{3}(48\pi) \\ &= 16\pi \end{aligned}$$

Substitute $2\pi r$ for C , where $r = 24$.

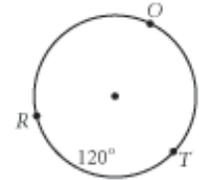
Simplify.

The arc length is 16π cm, or approximately 50.3 cm.



EXAMPLE C

If the length of \widehat{ROT} is 116π meters, what is the radius of the circle?



Solution

$m\widehat{ROT} = 240^\circ$, so \widehat{ROT} is $\frac{240}{360}$, or $\frac{2}{3}$ of the circumference.

$$116\pi = \frac{2}{3}C$$

Apply the Arc Length Conjecture.

$$116\pi = \frac{2}{3}(2\pi r)$$

Substitute $2\pi r$ for C .

$$348\pi = 4\pi r$$

Multiply both sides by 3.

$$87 = r$$

Divide both sides by 4π .

The radius is 87 m.



EXERCISES

You will need

For Exercises 1–8, state your answers in terms of



A calculator for Exercises 9–14

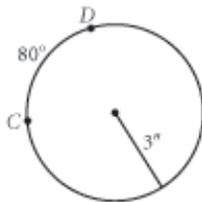


Construction tools for Exercise 18

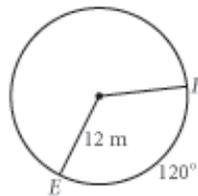


Geometry software for Exercises 18 and 24

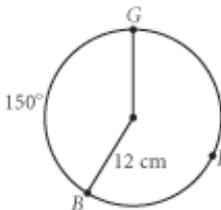
1. Length of \widehat{CD} is $\underline{\hspace{1cm}}$.



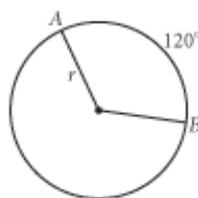
2. Length of \widehat{EF} is $\underline{\hspace{1cm}}$.



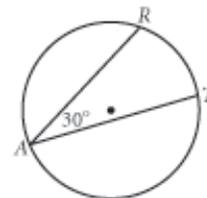
3. Length of \widehat{BIG} is $\underline{\hspace{1cm}}$.



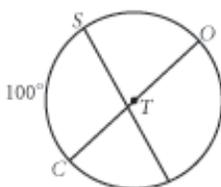
4. Length of \widehat{AB} is 6π m. The radius is $\underline{\hspace{1cm}}$.



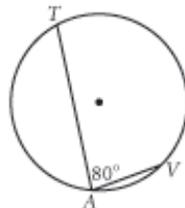
5. The radius is 18 ft. Length of \widehat{RT} is $\underline{\hspace{1cm}}$.



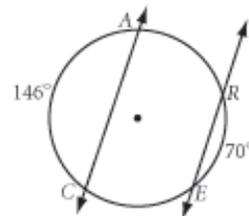
6. The radius is 9 m. Length of \widehat{SO} is $\underline{\hspace{1cm}}$.



7. Length of \widehat{TV} is 12π in. The diameter is $\underline{\hspace{1cm}}$.



8. Length of \widehat{AR} is 40π cm. $\widehat{CA} \parallel \widehat{RE}$. The radius is $\underline{\hspace{1cm}}$.

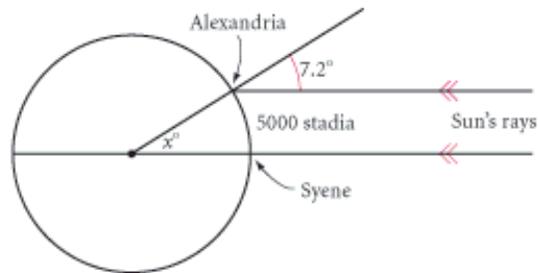


9. A go-cart racetrack has 100-meter straightaways and semicircular ends with diameters of 40 meters. Calculate the average speed in meters per minute of a go-cart if it completes 4 laps in 6 minutes. Round your answer to the nearest m/min. 
10. Astronaut Polly Hedra circles Earth every 90 minutes in a path above the equator. If the diameter of Earth is approximately 8000 miles, what distance along the equator will she pass directly over while eating a quick 15-minute lunch?
11. **Application** The Library of Congress reading room has desks along arcs of concentric circles. If an arc on the outermost circle with eight desks is about 12 meters long and makes up $\frac{1}{9}$ of the circle, how far are these desks from the center of the circle? How many desks would fit along an arc with the same central angle, but that is half as far from the center? Explain. 



The Library of Congress, Washington, D.C.

12. A Greek mathematician who lived in the 3rd century B.C.E., Eratosthenes, devised a clever method to calculate the circumference of Earth. He knew that the distance between Aswan (then called Syene) and Alexandria was 5000 Greek stadia (a stadium was a unit of distance at that time), or about 500 miles. At noon of the summer solstice, the Sun cast no shadow on a vertical pole in Syene, but at the same time in Alexandria a vertical pole did cast a shadow. Eratosthenes found that the angle between the vertical pole and the ray from the tip of the pole to the end of the shadow was 7.2° . From this he was able to calculate the ratio of the distance between the two cities to the circumference of Earth. Use this diagram to explain Eratosthenes' method, then use it to calculate the circumference of Earth in miles.

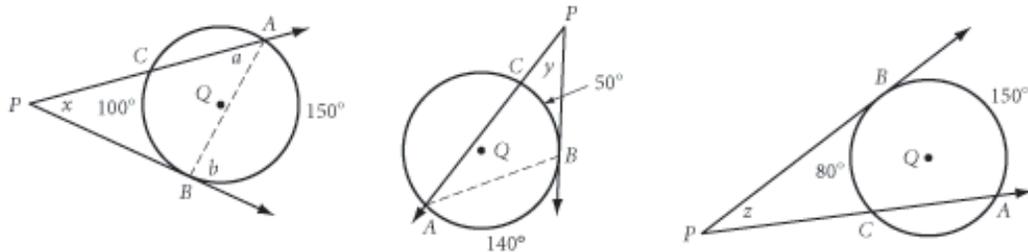


Review

13. **Angular velocity** is a measure of the rate at which an object revolves around an axis, and can be expressed in degrees per second. Suppose a carousel horse completes a revolution in 20 seconds. What is its angular velocity? Would another horse on the carousel have a different angular velocity? Why or why not?

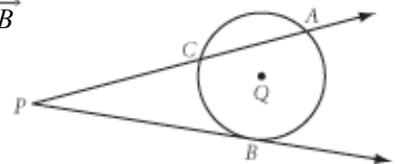


14. **Tangential velocity** is a measure of the distance an object travels along a circular path in a given amount of time. Like speed, it can be expressed in meters per second. Suppose two carousel horses complete a revolution in 20 seconds. The horses are 8 m and 6 m from the center of the carousel, respectively. What are the tangential velocities of the two horses? Round your answers to the nearest 0.1 m/s. Explain why the horses have equal angular velocities but different tangential velocities.
15. **Mini-Investigation** Use what you learned about the angle formed by a tangent and a chord in Lesson 6.4, Exercise 8, as well as what you know about inscribed angles and exterior angles of a triangle to find the missing angle measures in each diagram. Examine these cases to find a relationship between the measure of the angle formed by a tangent and a secant to a circle, $\angle BPA$, and the measures of the two intercepted arcs, \overline{AB} and \overline{BC} . Then copy and complete the conjecture below.

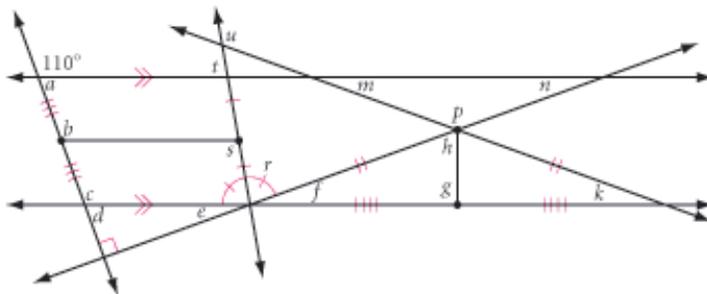


Conjecture: The measure of the angle formed by an intersecting tangent and secant to a circle is $\frac{1}{2}$ (Tangent-Secant Conjecture)

16. **Developing Proof** Given circle Q with secant \overrightarrow{PA} and tangent \overrightarrow{PB} in the diagram at right, prove the conjecture you made in the last exercise. Start by drawing auxiliary line \overline{AB} (h)



17. Calculate the measure of each lettered angle. (h)



Art

CONNECTION

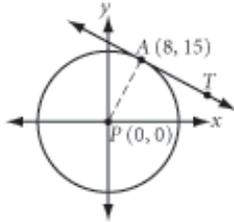
The traceries surrounding rose windows in Gothic cathedrals were constructed with only arcs and straight lines. The photo at right shows a rose window from Reims cathedral, which was built in the 13th century in Reims, a city in northeastern France. The overlaid diagram shows its constructions.



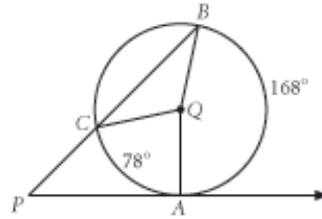
18. **Construction** Read the Art Connection above. Reproduce the constructions shown with your compass and straightedge or with geometry software. (h)

19. Find the measure of the angle formed by a clock's hands at 10:20. 

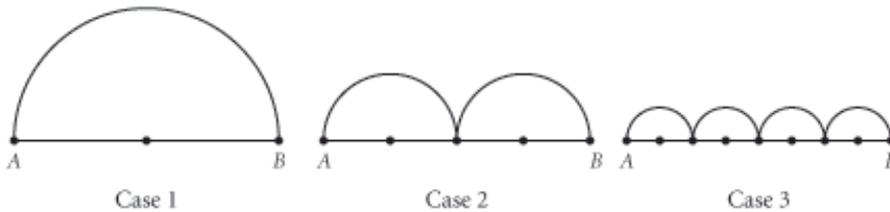
20. Circle P is centered at the origin. \overleftrightarrow{AT} is tangent to circle P at $A(8, 15)$. Find the equation of \overleftrightarrow{AT} .



21. \overline{PA} is tangent to circle Q . The line containing chord \overline{CB} passes through P . Find $m\angle P$.

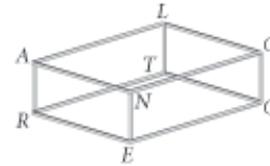


22. If $AB = 16$ cm, find the sum of the lengths of the semicircular bumps for each case. If the pattern continues, what would be the sum of the lengths of the bumps for Case 10?



23. How many different 3-edge routes are possible from R to G along the wire frame shown? 

24. **Technology** Use geometry software to pick any three points. Construct an arc through all three points. (Can it be done?) How do you find the center of the circle that passes through all three points?



project

RACETRACK GEOMETRY

If you had to start and finish at the same line of a racetrack, which lane would you choose? The inside lane has an obvious advantage. For a race to be fair, runners in the outside lanes must be given head starts, as shown in the photo.

Design a four-lane oval track with straightaways and semicircular ends. Show start and finish lines so that an 800-meter race can be run fairly in all four lanes. The semicircular ends must have inner diameters of 50 meters. The distance of one lap in the inner lane must be 800 meters.

Your project should contain

- ▶ A detailed drawing with labeled lengths.
- ▶ An explanation of the part that radius, lane width, and straightaway length plays in the design.



Exploration

Intersecting Lines Through a Circle

This chapter had five mini-investigations that explored the relationships between the angles and arcs formed by intersecting lines through a circle. Each mini-investigation resulted in a conjecture.

Lesson 6.2, Exercise 23, investigated the angle formed by intersecting tangents.

Lesson 6.4, Exercise 8, investigated the angle formed by the intersection of a tangent and a chord.

Lesson 6.5, Exercise 16, investigated the angle formed by intersecting chords.

Lesson 6.6, Exercise 9, investigated the angle formed by intersecting secants.

Lesson 6.7, Exercise 15, investigated the angle formed by the intersection of a secant and a tangent.

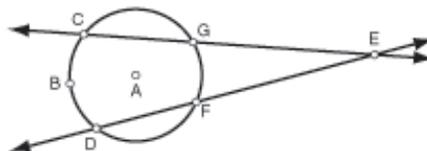
You will use Sketchpad to investigate these conjectures and how they are related.

Activity 1

Exploring Secants and Chords

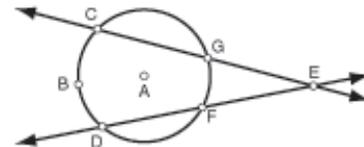
You will use Sketchpad to investigate intersecting secants and chords. Is there one unifying relationship that can explain both of these cases?

- Step 1 Construct a circle with center A and radius point B . Move point B to the left of point A .
- Step 2 Construct points C and D on the circle, and point E outside the circle. Move point C to the upper-left portion of the circle and point D to the lower-left portion of the circle. Move point E to the right of the circle.
- Step 3 Construct the lines \overleftrightarrow{CE} and \overleftrightarrow{DE} . Construct point F where \overleftrightarrow{DE} intersects the circle and point G where \overleftrightarrow{CE} intersects the circle. Your sketch should now look similar to the diagram below.



- Step 4 Measure $\angle CED$, the angle formed by the intersecting lines. You will observe this angle throughout the rest of this activity.
- Step 5 Measure \widehat{CBD} , the larger arc formed by the intersecting lines, by selecting points C, B, D , and circle A , in order, and choosing **Arc Angle** from the Measure menu.
- Step 6 Measure \widehat{FG} , the smaller arc formed by the intersecting lines, by selecting points F, G , and circle A , in order, and choosing **Arc Angle** again.
- Step 7 Investigate what happens as you move point E about the screen. Do you notice any patterns in the three measured values?
- Step 8 What happens if you drag point E to the edge of the circle? What happens if you drag it to the center of the circle? Which conjectures or definitions do these two cases illustrate?
- Step 9 Drag point E outside the circle again and to the right. Choose **Calculate** from the Measure menu to find the difference between the measures of the arcs, $m\widehat{CBD} - m\widehat{FG}$.
- Step 10 Drag points E, C , and D and look for patterns. What is the relationship between $m\angle CED$ and $m\widehat{CBD} - m\widehat{FG}$? Edit the calculation so that it gives the value of $m\angle CED$ and complete the conjecture.

$m\angle CED = 25.77^\circ$
 $m\widehat{CBD} \text{ on } \odot AB = 97.14^\circ$
 $m\widehat{FG} \text{ on } \odot AB = 45.60^\circ$
 $m\widehat{CBD} \text{ on } \odot AB - m\widehat{FG} \text{ on } \odot AB = 51.54^\circ$



Intersecting Secants Conjecture

The measure of an angle formed by two secants that intersect outside a circle is $\frac{?}{?}$.

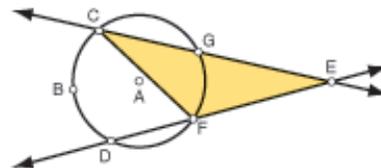
- Step 11 Drag point E inside the circle. Consider the two intersecting chords, \overline{CG} and \overline{DF} . Is the conjecture for intersecting secants also true for intersecting chords? Create a new calculation that gives the value of $m\angle CED$ and complete the conjecture.

Intersecting Chords Conjecture

The measure of an angle formed by two intersecting chords is $\frac{?}{?}$.

- Step 12 Drag point E outside and to the right of the circle. Construct \overline{CF} , and then construct $\triangle CEF$ by selecting the three vertices and choosing **Triangle Interior** from the Construct menu. How are $\angle CFD$ and $\angle FCG$ related to the intercepted arcs? How are they related to $\angle CED$? Use these relationships to explain why the Intersecting Secants Conjecture is true.

$m\angle CED = 27.79^\circ$
 $m\widehat{CBD} \text{ on } \odot AB = 117.77^\circ$
 $m\widehat{FG} \text{ on } \odot AB = 62.18^\circ$



Step 13 Drag point E inside the circle and observe what happens to the triangle. How is $\angle CED$ related to $\triangle CEF$ when point E is inside the circle? Explain why the Intersecting Chords Conjecture is different from the Intersecting Secants Conjecture.

Step 14 To find a universal calculation for both intersecting secants and intersecting chords, choose **Preferences** from the Edit menu and change the Units for Angle to **directed degrees**. Directed degrees are defined as positive for counterclockwise angle or arc measures and negative for clockwise measures. Drag point E in and out of the circle to see if the calculation for intersecting secants now works for intersecting chords. Can you explain why?



These circles in the sand were created by the wind blowing blades of grass.

Activity 2

Exploring Tangents

If you move a secant so that it intersects the circle at only one point, it becomes a tangent. You will continue to use the sketch from the previous activity to investigate what happens when the secants become tangents.

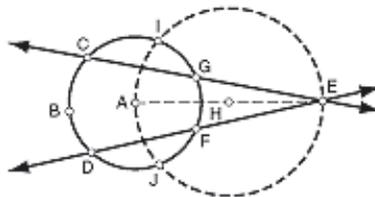
Step 1 Hide \overline{CF} and the triangle interior from Activity 1. Choose **Preferences** and change the Units for Angle back to **degrees**.

Step 2 Drag point E outside and to the right of the circle. Construct \overline{AE} with midpoint H . Construct circle H with radius point A . Label the two points of intersection of the circles I and J , and then hide circle H , \overline{AE} , and midpoint H .

$$m\angle CED = 23.05^\circ$$

$$m\widehat{CBD} \text{ on } \odot AB = 92.71^\circ$$

$$m\widehat{FG} \text{ on } \odot AB = 46.80^\circ$$



- Step 3 | Select points C and I in order. Choose **Action Buttons** from the Edit menu, click on **Movement** in the submenu, and press OK in the dialog box. Press the new $Move\ C \rightarrow I$ button. Does the calculation for intersecting secants still work? Complete the conjecture.

Tangent-Secant Conjecture

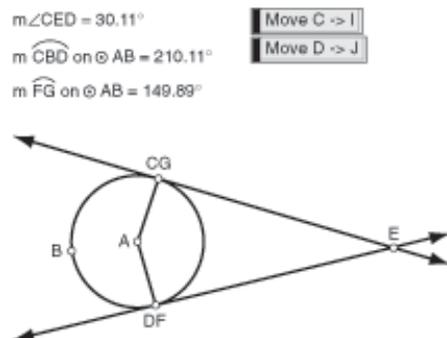
The measure of an angle formed by an intersecting tangent and secant to a circle is $\frac{1}{2}$.

- Step 4 | Repeat the process in the last step to create a $Move\ D \rightarrow J$ button and press it to get two tangents. Check that the calculation for intersecting secants still works, and then look for a new relationship. How is $m\angle CED$ related to $m\widehat{FG}$? Use this relationship to complete the conjecture.

Intersecting Tangents Conjecture

The measure of an angle formed by intersecting tangents to a circle is $\frac{1}{2}$.

- Step 5 | Construct the radii from A to the two points of tangency. Use the Quadrilateral Sum Conjecture and the Tangent Conjecture to explain why the Intersecting Tangents Conjecture is true. Then use your algebra skills to show why this relationship is equivalent to the calculation for intersecting secants.



- Step 6 | Delete the radii. Select point E and circle A , and choose **Merge Point To Circle** from the Edit menu. What happens to the measure of the smaller arc? Superimposing points D and E forms one tangent and one chord. Drag point D toward point E while observing the measures of \widehat{CBD} and $\angle CED$ and complete the conjecture.

Tangent-Chord Conjecture

The measure of an angle formed by the intersection of a tangent and chord at the point of tangency is $\frac{1}{2}$.

- Step 7 | As an extension, explain why the construction in Step 2 determines the points of tangency from point E to circle A .

CHAPTER 6 REVIEW

In this chapter you learned some new circle vocabulary and solved real-world application problems involving circles. You discovered the relationship between a radius and a tangent line. You discovered special relationships between angles and their intercepted arcs. And you learned about the special ratio π and how to use it to calculate the circumference of a circle and the length of an arc.

You should be able to sketch these terms from memory: *chord*, *tangent*, *central angle*, *inscribed angle*, and *intercepted arc*. And you should be able to explain the difference between arc measure and arc length.



EXERCISES

You will need



Construction tools
for Exercises 21–24, 67,
and 70

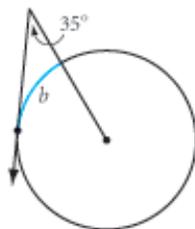


A calculator
for Exercises 11, 12, and 27–33

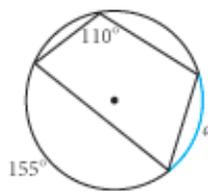
1. What do you think is the most important or useful circle property you learned in this chapter? Why?
2. How can you find the center of a circle with a compass and a straightedge? With patty paper? With the right-angled corner of a carpenter's square?
3. What does the path of a satellite have to do with the Tangent Conjecture?
4. Explain the difference between the degree measure of an arc and its arc length.

Solve Exercises 5–19. If the exercise uses the “=” sign, answer in terms of π . If the exercise uses the “ \approx ” sign, give your answer accurate to one decimal place.

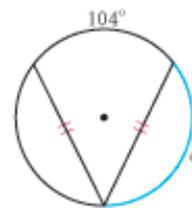
5. $b = ?$



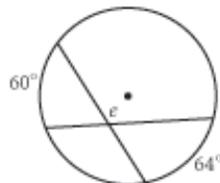
6. $a = ?$



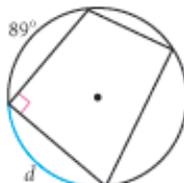
7. $c = ?$



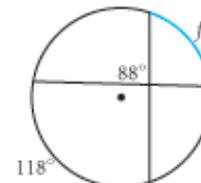
8. $e = ?$



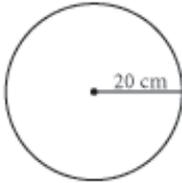
9. $d = ?$



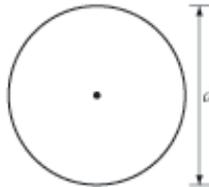
10. $f = ?$



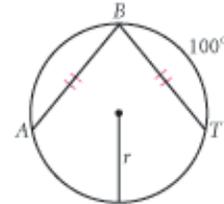
11. circumference \approx ?



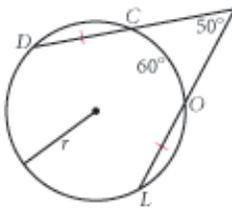
12. circumference = 132 cm
 $d \approx$?



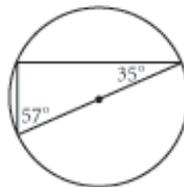
13. $r = 27$ cm. The length of \overline{AB} is ?.



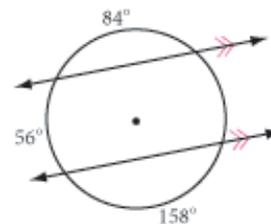
14. $r = 36$ ft. The length of \overline{CD} is ?.



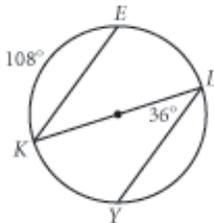
15. **Developing Proof** What's wrong with this picture?



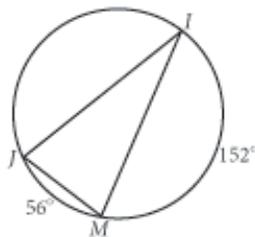
16. **Developing Proof** What's wrong with this picture?



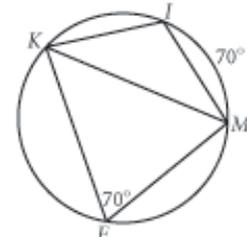
17. **Developing Proof** Explain why $\overline{KE} \parallel \overline{YL}$.



18. **Developing Proof** Explain why $\triangle JIM$ is isosceles.



19. **Developing Proof** Explain why $\triangle KIM$ is isosceles.

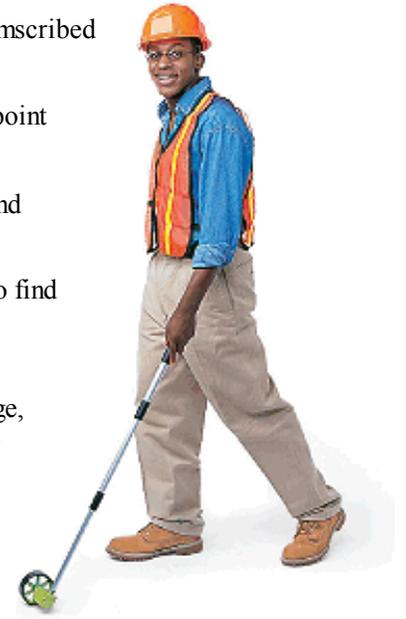


20. On her latest archaeological dig, Ertha Diggs has unearthed a portion of a cylindrical column. All she has with her is a pad of paper. How can she use it to locate the diameter of the column?

21. **Construction** Construct a scalene obtuse triangle. Construct the circumscribed circle.
22. **Construction** Construct a scalene acute triangle. Construct the inscribed circle.
23. **Construction** Construct a rectangle. Is it possible to construct the circumscribed circle, the inscribed circle, neither, or both?

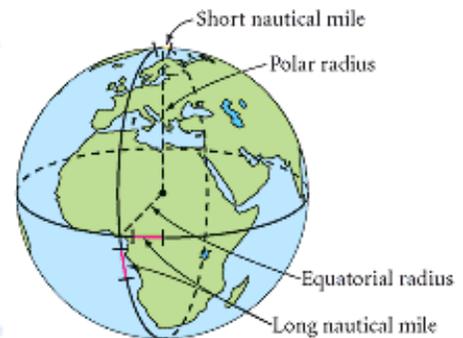


24. **Construction** Construct a rhombus. Is it possible to construct the circumscribed circle, the inscribed circle, neither, or both?
25. Find the equation of the line tangent to circle S centered at $(1, 1)$ if the point of tangency is $(5, 4)$.
26. Find the center of the circle passing through the points $(-7, 5)$, $(0, 6)$, and $(1, -1)$.
27. Rashid is an apprentice on a road crew for a civil engineer. He needs to find a trundle wheel similar to but larger than the one shown at right. If each rotation is to be 1 m, what should be the diameter of the trundle wheel?
28. Melanie rides the merry-go-round on her favorite horse on the outer edge, 8 meters from the center of the merry-go-round. Her sister, Melody, sits in the inner ring of horses, 3 meters in from Melanie. In 10 minutes, they go around 30 times. What is the average speed of each sister?
29. Read the Geography Connection below. Given that the polar radius of Earth is 6357 kilometers and that the equatorial radius of Earth is 6378 kilometers, use the original definition to calculate one nautical mile near a pole and one nautical mile near the equator. Show that the international nautical mile is between both values.



Geography CONNECTION

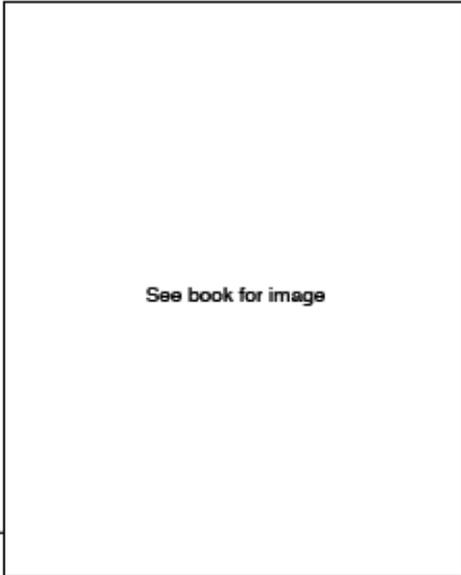
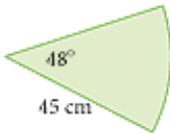
One nautical mile was originally defined to be the length of one minute of arc of a great circle of Earth. (A great circle is the intersection of the sphere and a plane that cuts through its center. There are 60 minutes of arc in each degree.) But Earth is not a perfect sphere. It is wider at the great circle of the equator than it is at the great circle through the poles. So defined as one minute of arc, one nautical mile could take on a range of values. To remedy this, an international nautical mile was defined as 1.852 kilometers (about 1.15 miles).



30. While talking to his friend Tara on the phone, Dmitri sees a lightning flash, and 5 seconds later he hears thunder. Two seconds after that, Tara, who lives 1 mile away, hears it. Sound travels at 1100 feet per second. Draw and label a diagram showing the possible locations of the lightning strike.
31. King Arthur wishes to seat all his knights at a round table. He instructs Merlin to design and create an oak table large enough to seat 100 people. Each knight is to have 2 ft along the edge of the table. Help Merlin calculate the diameter of the table.



32. If the circular moat should have been a circle of radius 10 meters instead of radius 6 meters, how much greater should the larger moat's circumference have been?
33. The part of a circle enclosed by a central angle and the arc it intercepts is called a **sector**. The sector of a circle shown below can be curled into a cone by bringing the two straight 45-cm edges together. What will be the diameter of the base of the cone?

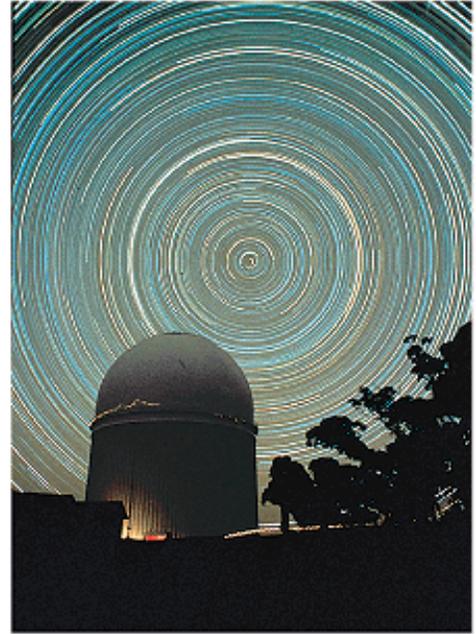


MIXED REVIEW

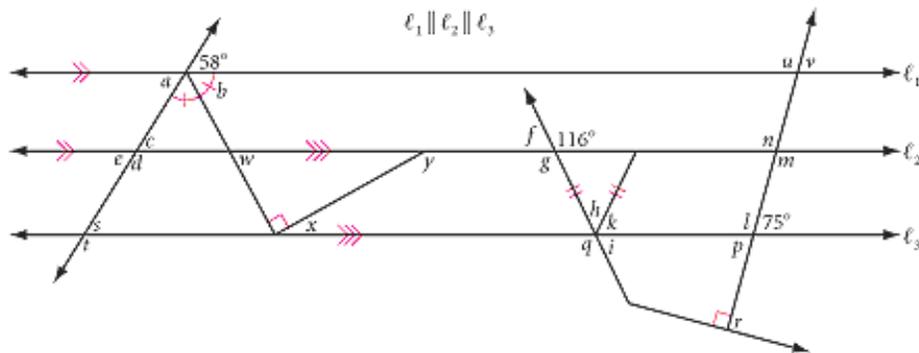
In Exercises 34–56, identify the statement as true or false. For each false statement, explain why it is false or sketch a counterexample.

34. If a triangle has two angles of equal measure, then the third angle is acute.
35. If two sides of a triangle measure 45 cm and 36 cm, then the third side must be greater than 9 cm and less than 81 cm.
36. The diagonals of a parallelogram are congruent.
37. The measure of each angle of a regular dodecagon is 150° .
38. The perpendicular bisector of a chord of a circle passes through the center of the circle.
39. If \overline{CD} is the midsegment of trapezoid $PLYR$ with \overline{PL} one of the bases, then $CD = \frac{1}{2}(PL + YR)$.
40. In $\triangle BOY$, $BO = 36$ cm, $m\angle B = 42^\circ$, and $m\angle O = 28^\circ$. In $\triangle GRL$, $GR = 36$ cm, $m\angle R = 28^\circ$, and $m\angle L = 110^\circ$. Therefore, $\triangle BOY \cong \triangle GRL$.
41. If the sum of the measures of the interior angles of a polygon is less than 1000° , then the polygon has fewer than seven sides.
42. The sum of the measures of the three angles of an obtuse triangle is greater than the sum of the measures of the three angles of an acute triangle.
43. The sum of the measures of one set of exterior angles of a polygon is always less than the sum of the measures of interior angles.

44. Both pairs of base angles of an isosceles trapezoid are supplementary.
45. If the base angles of an isosceles triangle each measure 48° , then the vertex angle has a measure of 132° .
46. Inscribed angles that intercept the same arc are supplementary.
47. The measure of an inscribed angle in a circle is equal to the measure of the arc it intercepts.
48. The diagonals of a rhombus bisect the angles of the rhombus.
49. The diagonals of a rectangle are perpendicular bisectors of each other.
50. If a triangle has two angles of equal measure, then the triangle is equilateral.
51. If a quadrilateral has three congruent angles, then it is a rectangle.
52. In two different circles, arcs with the same measure are congruent.
53. The ratio of the diameter to the circumference of a circle is π .
54. If the sum of the lengths of two consecutive sides of a kite is 48 cm, then the perimeter of the kite is 96 cm.
55. If the vertex angles of a kite measure 48° and 36° , then the nonvertex angles each measure 138° .
56. All but seven statements in Exercises 34–56 are false.
57. Find the measure of each lettered angle in the diagram below.

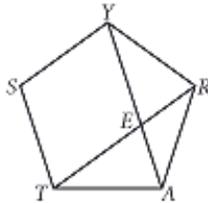


The concentric circles in the sky are actually a time exposure photograph of the movement of the stars in a night.

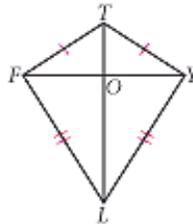


Developing Proof In Exercises 58–60, from the information given, determine which triangles, if any, are congruent. State the congruence conjecture that supports your congruence statement.

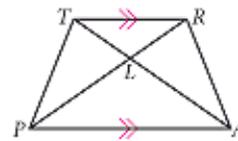
58. *STARY* is a regular pentagon.



59. *FLYT* is a kite.



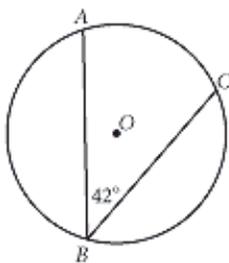
60. *PART* is an isosceles trapezoid.



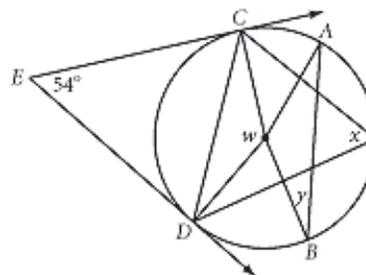
61. Adventurer Dakota Davis has uncovered a piece of triangular tile from a mosaic. A corner is broken off. Wishing to repair the mosaic, he lays the broken tile piece down on paper and traces the straight edges. With a ruler he then extends the unbroken sides until they meet. What triangle congruence shortcut guarantees that the tracing reveals the original shape?



62. Circle *O* has a radius of 24 inches. Find the measure and the length of \overline{AC} .



63. **Developing Proof** \overline{EC} and \overline{ED} are tangent to the circle, and $AB = CD$. Find the measure of each lettered angle. Explain.



64. Use your protractor to draw and label a pair of supplementary angles that is not a linear pair.

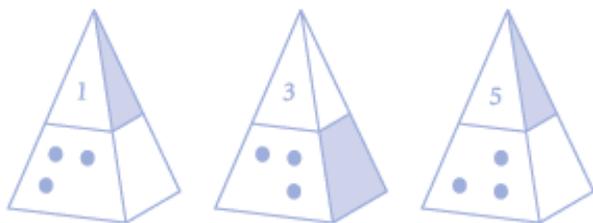
65. Find the function rule $f(n)$ of this sequence and find the 20th term.

n	1	2	3	4	5	6	...	n	...	20
$f(n)$	5	1	-3	-7	-11	-15	

66. The design at right shows three hares joined by three ears, although each hare appears to have two ears of its own.
- Does the design have rotational symmetry?
 - Does the design have reflectional symmetry?
67. **Construction** Construct a rectangle whose length is twice its width.
68. If $AB = 15$ cm, C is the midpoint of \overline{AB} , D is the midpoint of \overline{AC} , and E is the midpoint of \overline{DC} , what is the length of \overline{EB} ?
69. Draw the next shape in this pattern.



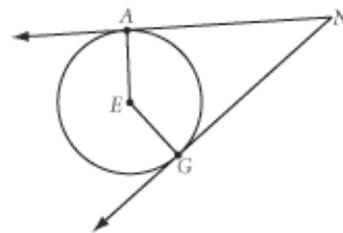
Private Collection, Berkeley, California
Ceramist, Diana Hall



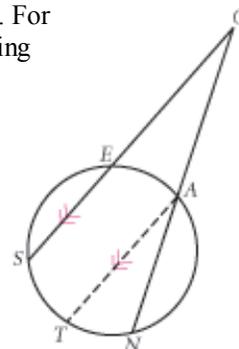
70. **Construction** Construct any triangle. Then construct its centroid.

TAKE ANOTHER LOOK

- Developing Proof** Show how the Tangent Segments Conjecture follows logically from the Tangent Conjecture and the converse of the Angle Bisector Conjecture.
- Developing Proof** Investigate the quadrilateral formed by two tangent segments to a circle and the two radii to the points of tangency. State a conjecture. Explain why your conjecture is true, based on the properties of radii and tangents.
- Developing Proof** State the Cyclic Quadrilateral Conjecture in “if-then” form. Then state the converse of the conjecture in “if-then” form. Is the converse also true?
- Developing Proof** A quadrilateral that *can* be inscribed in a circle is also called a cyclic quadrilateral. Which of these quadrilaterals are always cyclic: parallelograms, kites, isosceles trapezoids, rhombuses, rectangles, or squares? Which ones are never cyclic? Explain why each is or is not always cyclic.
- Use graph paper or a graphing calculator to graph the data collected from the investigation in Lesson 6.5. Graph the diameter on the x -axis and the circumference on the y -axis. What is the slope of the best-fit line through the data points? Does this confirm the Circumference Conjecture? Explain.



6. **Construction** Construct a tangent to a circle from a point external to the circle. For this to be a construction, you need to determine the precise point of tangency using your construction tools, not just by placing your ruler on the point and circle so that it appears to be tangent. (*Hint:* Look at the diagram on page 357.)
7. **Developing Proof** In Lesson 6.6, Exercise 10, you proved the Intersecting Secants Conjecture using auxiliary line \overline{SA} and the Triangle Exterior Angle Conjecture. Another proof uses an auxiliary line parallel to secant \overline{SE} through point A and the Parallel Lines Intercepted Arcs Conjecture. Write this proof.



Assessing What You've Learned

With the different assessment methods you've used so far, you should be getting the idea that assessment means more than a teacher giving you a grade. All the methods presented so far could be described as self-assessment techniques. Many are also good study habits. Being aware of your *own* learning and progress is the best way to stay on top of what you're doing and to achieve the best results.



WRITE IN YOUR JOURNAL

- ▶ You may be at or near the end of your school year's first semester. Look back over the first semester and write about your strengths and needs. What grade would you have given yourself for the semester? How would you justify that grade?
- ▶ Set new goals for the new semester or for the remainder of the year. Write them in your journal and compare them to goals you set at the beginning of the year. How have your goals changed? Why?



ORGANIZE YOUR NOTEBOOK Review your notebook and conjectures list to be sure they are complete and well organized. Write a one-page chapter summary.



UPDATE YOUR PORTFOLIO Choose a piece of work from this chapter to add to your portfolio. Document the work according to your teacher's instructions.



PERFORMANCE ASSESSMENT While a classmate, a friend, a family member, or a teacher observes, carry out one of the investigations or Take Another Look activities from this chapter. Explain what you're doing at each step, including how you arrived at the conjecture.



WRITE TEST ITEMS Divide the lessons from this chapter among group members and write at least one test item per lesson. Try out the test questions written by your classmates and discuss them.



GIVE A PRESENTATION Give a presentation on an investigation, exploration, Take Another Look project, or puzzle. Work with your group, or try giving a presentation on your own.