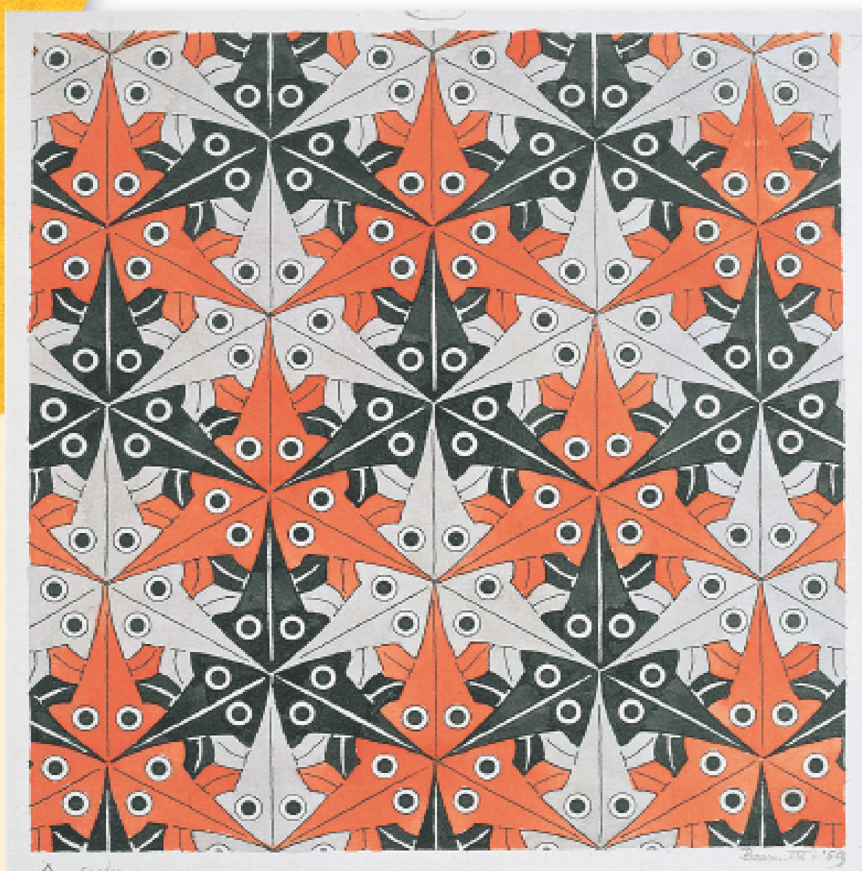
CHAPTER

4

Discovering and Proving Triangle Properties



Is it possible to make a representation of recognizable figures that has no background?

M. C. ESCHER

Symmetry Drawing E103, M. C. Escher, 1959

OBJECTIVES

In this chapter you will

- learn why triangles are so useful in structures
- discover relationships between the sides and angles of triangles
- learn about the conditions that guarantee that two triangles are congruent

LESSON 4.1

Triangle Sum Conjecture

Teaching is the art of assisting discovery.

ALBERT VAN DOREN

Triangles have certain properties that make them useful in all kinds of structures, from bridges to high-rise buildings. One such property of triangles is their rigidity. If you build shelves like the first set shown at right, they will sway. But if you nail another board at the diagonal as in the second set, creating a triangle, you will have rigid shelves.

Another application of triangles is a procedure used in surveying called **triangulation.** This procedure allows surveyors to locate points or positions on a map by measuring angles and distances and creating a network of triangles. Triangulation is based on an important property of plane geometry that you will discover in this lesson.

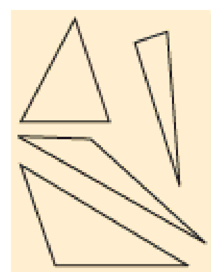
Investigation

The Triangle Sum

You will need

- a protractor
- a straightedge
- scissors
- patty paper

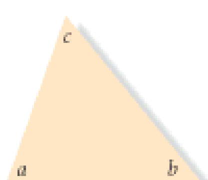
There are an endless variety of triangles that you can draw, with different shapes and angle measures. Do their angle measures have anything in common? Start by drawing different kinds of triangles. Make sure your group has at least one acute and one obtuse triangle.



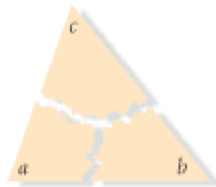
Step 1 Measure the three angles of each triangle as accurately as possible with your protractor.

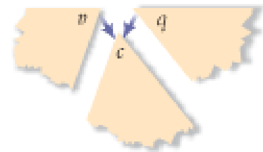
Step 2 Find the sum of the measures of the three angles in each triangle. Compare results with others in your group. Does everyone get about the same result? What is it?

Step 3 Check the sum another way. Write the letters *a, b,* and *c* in the interiors of the three angles of one of the triangles, and carefully cut out the triangle.



Step 4 Tear off the three angles. Arrange them so that their vertices meet at a point. How does this arrangement show the sum of the angle measures?





Step 5 Compare results with others in your group. State your observations as a conjecture.

Triangle Sum Conjecture C-17

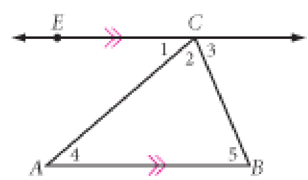
The sum of the measures of the angles in every triangle is ?.

Developing Proof The investigation may have convinced you that the Triangle Sum Conjecture is true, but can you explain *why* it is true for every triangle?

As a group, explain why the Triangle Sum Conjecture is true by writing a **paragraph proof,** a deductive argument that uses written sentences to support its claims with reasons.

Another reasoning strategy you might use is to add an **auxiliary line,** an extra line or segment that helps with a proof. Your group may have formed an auxiliary line by rearranging the angles in the investigation. If you rotated $\angle A$ and $\angle B$ and left $\angle C$ pointing up, then how is the resulting line related to the original triangle? Draw any $\triangle ABC$ and draw in that auxiliary line.

The figure at right includes $\overleftrightarrow{EC}$, an auxiliary line parallel to side $\overline{AB}$. Use this diagram to discuss these questions with your group.



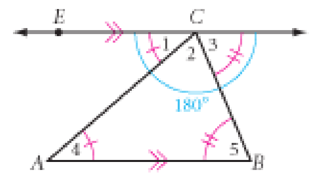
- What are you trying to prove?
- What is the relationship among $\angle 1$, $\angle 2$, and $\angle 3$?
- Why was the auxiliary line drawn to be parallel to one of the sides?
- What other congruencies can you determine from the diagram?

Use your responses to these questions to mark your diagram. Discuss how you can use the information you have to prove that the Triangle Sum Conjecture is true for every triangle. As a group, write a paragraph proof. When you are satisfied with your group's proof, compare it to the one presented on the next page.

Paragraph Proof: The Triangle Sum Conjecture

Consider $\angle 1$ and $\angle 2$ together as a single angle that forms a linear pair with $\angle 3$. By the Linear Pair Conjecture, their measures must add up to 180°.

$$m\angle 1 + m\angle 2 + m\angle 3 = 180°$$



$\overline{AC}$ and $\overline{BC}$ form transversals between parallel lines $\overleftrightarrow{EC}$ and $\overleftrightarrow{AB}$. By the Alternate Interior Angles Conjecture, 1 and 4 are congruent and 3 and 5 are congruent, so their measures are equal.

$$m\angle 1 = m\angle 4$$

$$m\angle 3 = m\angle 5$$

Substitute $m\angle 4$ for $m\angle 1$, and $m\angle 5$ for $m\angle 3$ in the first equation above.

$$m\angle 4 + m\angle 2 + m\angle 5 = 180°$$

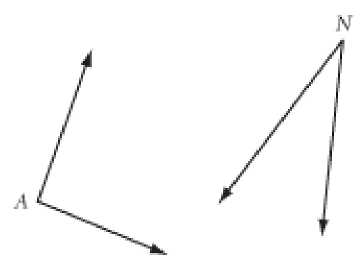
Therefore, the measures of the angles in a triangle add up to 180°. ■

So far, you have been writing deductive arguments to explain why conjectures are true. The paragraph proof format puts a little more emphasis on justifying your reasons. You will also learn about another proof format later in this chapter.

If you have two angles of a triangle, you can use the Triangle Sum Conjecture to construct the third angle. This example shows one way to do this.

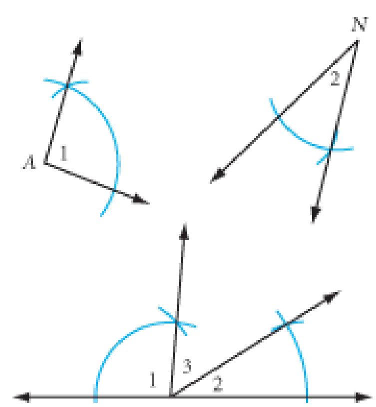
EXAMPLE

Given $\angle A$ and $\angle N$, construct $\angle D$, the third angle of $\triangle AND$.



► Solution

Label $\angle A$ and $\angle N$ as $\angle 1$ and $\angle 2$ respectively. Draw a line. Duplicate $\angle 1$ opening to the left on this line. Duplicate $\angle 2$ opening to the right at the same vertex on this line. Because the measures of the three angles add to 180°, the measure of $\angle 3$ is equal to that of $\angle D$.



EXERCISES

You will need

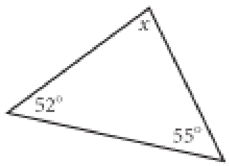
Geometry software for Exercise 1

Construction tools for Exercises 10–13

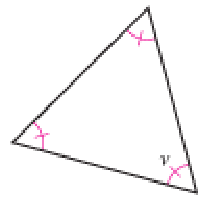
1. ***Technology*** Using geometry software, construct a triangle. Use the software to measure the three angles and calculate their sum. Drag the vertices and describe your observations.

Use the Triangle Sum Conjecture to determine each lettered angle measure in Exercises 2–5. You might find it helpful to copy the diagrams so you can write on them.

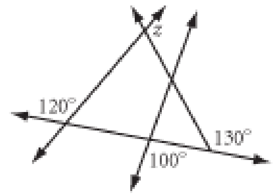
2. $x = \underline{?}$



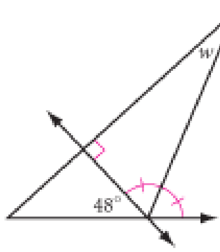
3. $v = \underline{?}$



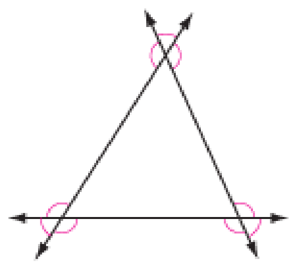
4. $z = \underline{?}$



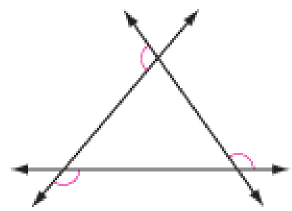
5. $w = \underline{?}$



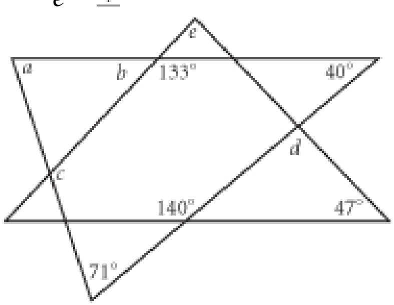
6. Find the sum of the measures of the marked angles.



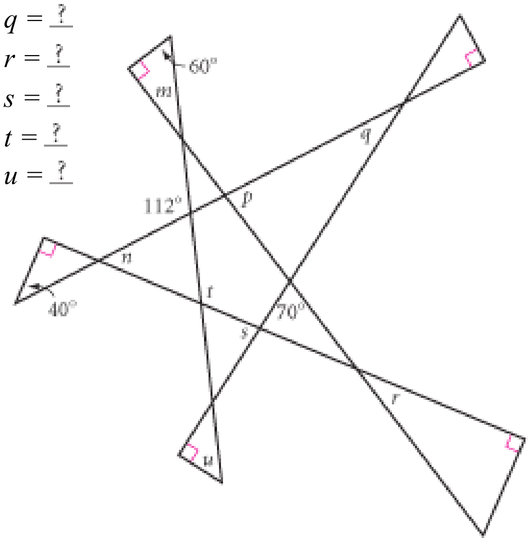
7. Find the sum of the measures of the marked angles.



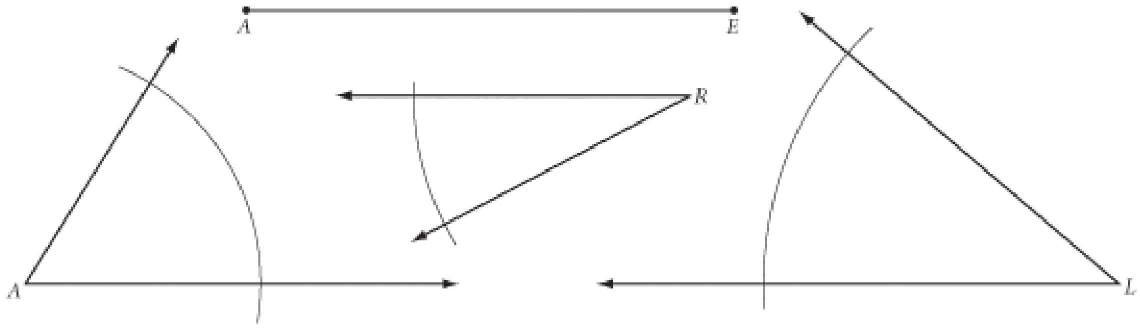
8. $a = \underline{?}$
$b = \underline{?}$
$c = \underline{?}$
$d = \underline{?}$
$e = \underline{?}$



9. $m = \underline{?}$
$n = \underline{?}$
$p = \underline{?}$
$q = \underline{?}$
$r = \underline{?}$
$s = \underline{?}$
$t = \underline{?}$
$u = \underline{?}$



In Exercises 10–12, use what you know to construct each figure. Use only a compass and a straightedge.

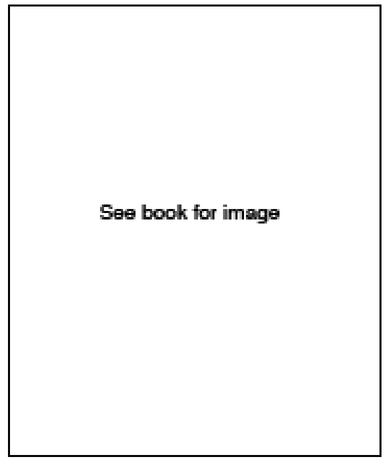


10. ***Construction*** Given $\angle A$ and $\angle R$ of $\triangle ARM$, construct $\angle M$.

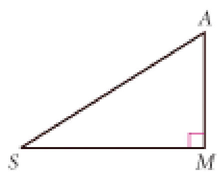
11. ***Construction*** In $\triangle LEG$, $m\angle E = m\angle G$. Given $\angle L$, construct $\angle G$. ⓗ

12. ***Construction*** Given $\angle A$, $\angle R$, and side $\overline{AE}$ construct $\triangle EAR$. ⓗ

13. ***Construction*** Repeat Exercises 10–12 with patty-paper constructions.



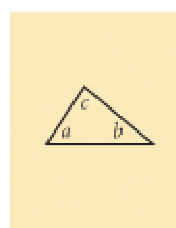
14. ***Developing Proof*** In $\triangle MAS$ below, $\angle M$ is a right angle. Let's call the two acute angles, $\angle A$ and $\angle S$, "wrong angles." Write a paragraph proof or use algebra to show that "two wrongs make a right," at least for angles in a right triangle.



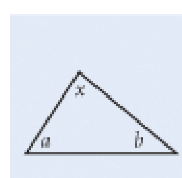
15. ***Developing Proof*** In your own words, prove the Triangle Sum Conjecture. What conjectures must we accept as true in order to prove it?

16. Use your ruler and protractor to draw $\triangle PDQ$ if $m\angle P = 40°$, $m\angle Q = 55°$, and $PD = 7$ cm. How can the Triangle Sum Conjecture make this easier to do?

17. ***Mini-Investigation*** Suppose two angles of one triangle have the same measures as two angles of another triangle. What can you conclude about the third pair of angles?



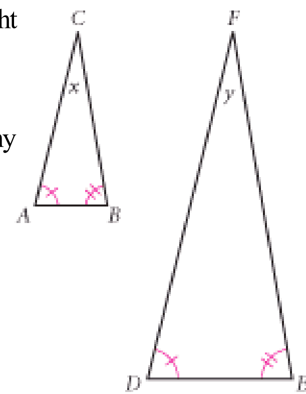
Draw a triangle on your notebook paper. Create a second triangle on patty paper by tracing two of the angles of your original triangle, but make the side between your new angles longer than the corresponding side in the original triangle. How do the third angles in the two triangles compare?



Conjecture: If two angles of one triangle are equal in measure to two angles of another triangle, then the third angles of the triangles __?__. (Third Angle Conjecture)

18. ***Developing Proof*** Use the Triangle Sum Conjecture and the figures at right to write a paragraph proof explaining why the Third Angle Conjecture is true. ⓗ

19. ***Developing Proof*** Write a paragraph proof, or use algebra, to explain why each angle of an equiangular triangle measures 60°.



Review

In Exercises 20–24, tell whether the statement is true or false. For each false statement, explain why it is false or sketch a counterexample.

20. If two sides in one triangle are congruent to two sides in another triangle, then the two triangles are congruent.

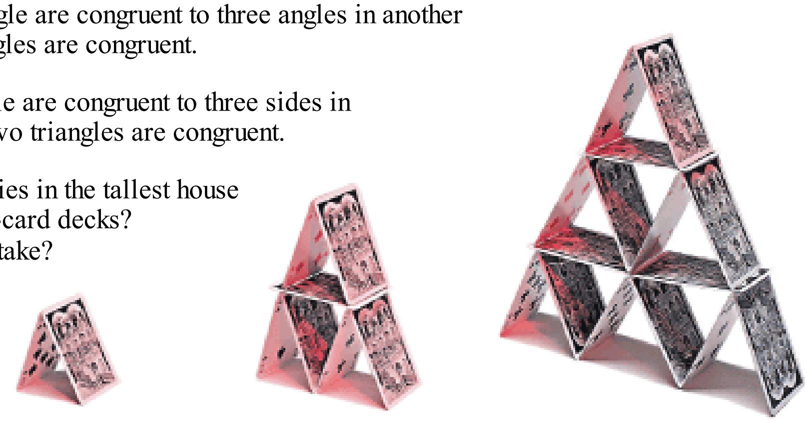
21. If two angles in one triangle are congruent to two angles in another triangle, then the two triangles are congruent.

22. If a side and an angle in one triangle are congruent to a side and an angle in another triangle, then the two triangles are congruent.

23. If three angles in one triangle are congruent to three angles in another triangle, then the two triangles are congruent.

24. If three sides in one triangle are congruent to three sides in another triangle, then the two triangles are congruent.

25. What is the number of stories in the tallest house you can build with two 52-card decks? How many cards would it take?



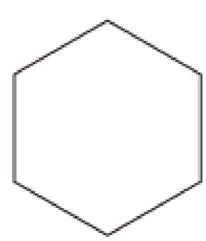
One story (2 cards) Two stories (7 cards) Three stories (15 cards)

IMPROVING YOUR VISUAL THINKING SKILLS

Dissecting a Hexagon I

Trace this regular hexagon twice.

1. Divide one hexagon into four congruent trapezoids.
2. Divide the other hexagon into eight congruent parts. What shape is each part?



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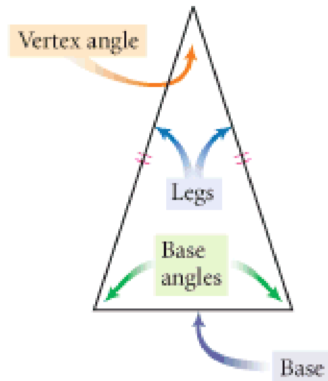
LESSON

4.2

Properties of Isosceles Triangles

Imagination is built upon knowledge.
ELIZABETH STUART PHELPS

Recall from Chapter 1 that an isosceles triangle is a triangle with at least two congruent sides. In an isosceles triangle, the angle between the two congruent sides is called the vertex angle, and the other two angles are called the base angles. The side between the two base angles is called the base of the isosceles triangle. The other two sides are called the **legs.**



In this lesson you'll discover some properties of isosceles triangles.

The famous Transamerica Building in San Francisco contains many isosceles triangles.

The Rock and Roll Hall of Fame and Museum structure is a pyramid containing many triangles that are isosceles *and* equilateral.

Architecture CONNECTION

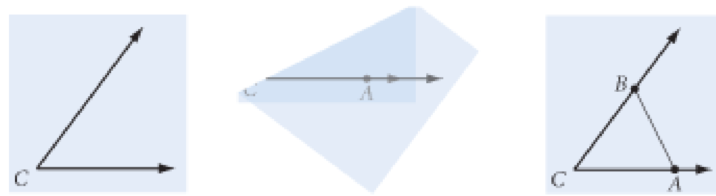
The Rock and Roll Hall of Fame and Museum in Cleveland, Ohio, is a dynamic structure. Its design reflects the innovative music that it honors. The front part of the museum is a large glass pyramid, divided into small triangular windows. The pyramid structure rests on a rectangular tower and a circular theater that looks like a performance drum. Architect I. M. Pei (b 1917) used geometric shapes to capture the resonance of rock and roll musical chords.

Investigation 1
Base Angles in an Isosceles Triangle

You will need

- patty paper
- a straightedge
- a protractor

Let's examine the angles of an isosceles triangle. Each person in your group should draw a different angle for this investigation. Your group should have at least one acute angle and one obtuse angle.



Step 1 Step 2 Step 3

Step 1 Draw an angle on patty paper. Label it $\angle C$. This angle will be the vertex angle of your isosceles triangle.

Step 2 Place a point A on one ray. Fold your patty paper so that the two rays match up. Trace point A onto the other ray.

Step 3 Label the point on the other ray point B. Draw $\overline{AB}$. You have constructed an isosceles triangle. Explain how you know it is isosceles. Name the base and the base angles.

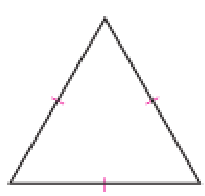
Step 4 Use your protractor to compare the measures of the base angles. What relationship do you notice? How can you fold the paper to confirm your conclusion?

Step 5 Compare results in your group. Was the relationship you noticed the same for each isosceles triangle? State your observations as your next conjecture.

Isosceles Triangle Conjecture C-18

If a triangle is isosceles, then $\underline{?}$.

Equilateral triangles have at least two congruent sides, so they fit the definition of isosceles triangles. That means any properties you discover for isosceles triangles will also apply to equilateral triangles. How does the Isosceles Triangle Conjecture apply to equilateral triangles?

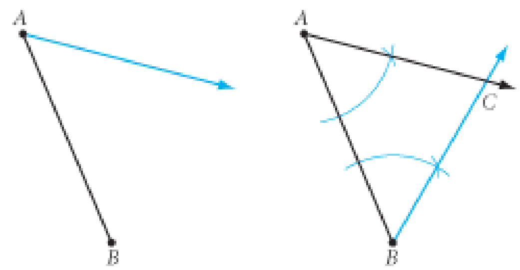


You can switch the "if" and "then" parts of the Isosceles Triangle Conjecture to obtain the converse of the conjecture. Is the converse of the Isosceles Triangle Conjecture true? Let's investigate.

Investigation 2
Is the Converse True?

You will need
- a compass
- a straightedge

Suppose a triangle has two congruent angles. Must the triangle be isosceles?



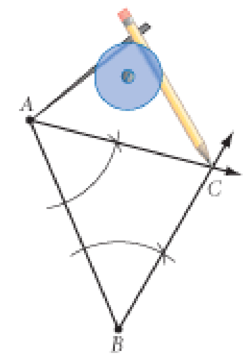
Step 1 Step 2

Step 1 Draw a segment and label it $\overline{AB}$. Draw an acute angle at point A. This angle will be a base angle. (Why can't you draw an obtuse angle as a base angle?)

Step 2 Copy $\angle A$ at point B on the same side of $\overline{AB}$ Label the intersection of the two rays point C.

Step 3 Use your compass to compare the lengths of sides $\overline{AC}$ and BC. What relationship do you notice? How can you use patty paper to confirm your conclusion?

Step 4 Compare results in your group. State your observation as your next conjecture.



Converse of the Isosceles Triangle Conjecture C-19

If a triangle has two congruent angles, then _?_ ..

EXERCISES

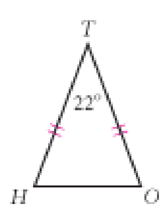
Keymath.com Practice Your Skills

You will need

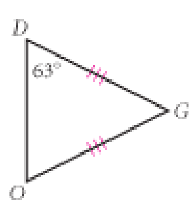
Construction tools for Exercises 15–17

For Exercises 1–6, use your new conjectures to find the missing measures.

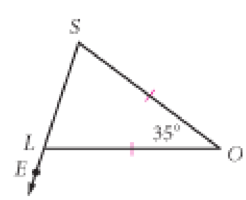
1. $m\angle H =$ _?_ (h)



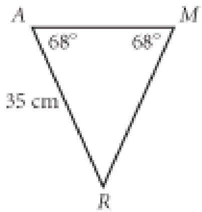
2. $m\angle G =$ _?_



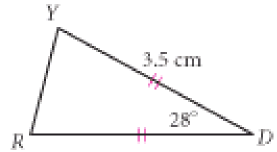
3. $m\angle OLE =$ _?_



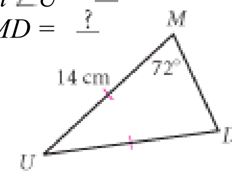
4. $m\angle R =$ _?_
$RM =$ _?_



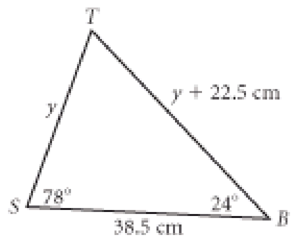
5. $m\angle Y =$ _?_
$RD =$ _?_



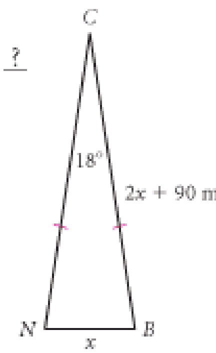
6. The perimeter of $\triangle MUD$ is 36.6 cm.
$m\angle D =$ _?_
$m\angle U =$ _?_
$MD =$ _?_



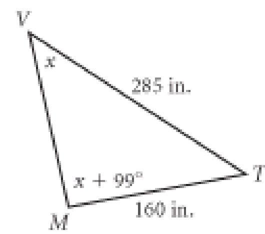
7. $m\angle T =$ _?_
perimeter of $\triangle TBS$ _?_



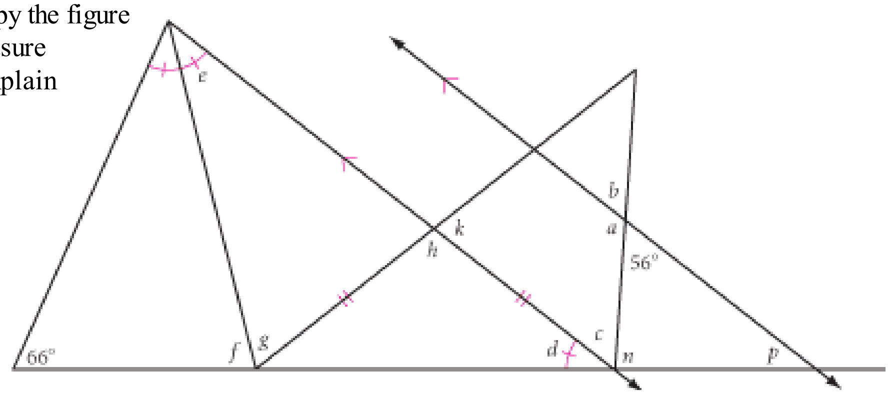
8. The perimeter of $\triangle NBC$ is 555 m.
$NB =$ _?_
$m\angle N =$ _?_



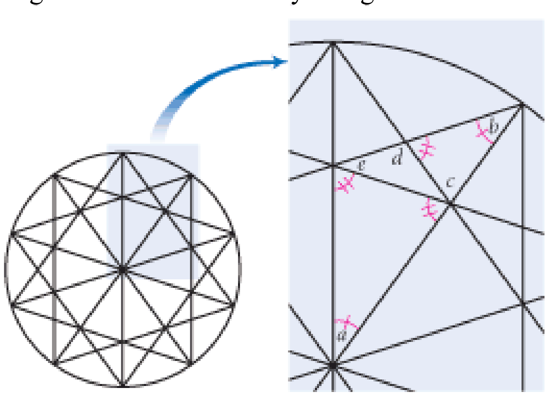
9. The perimeter of $\triangle MTV$ is 605 in.
$MV =$ _?_
$m\angle M =$ _?_

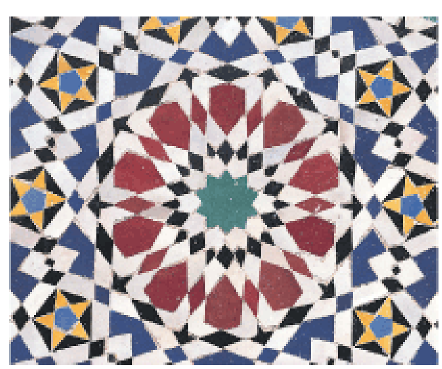


10. ***Developing Proof*** Copy the figure at right. Calculate the measure of each lettered angle. Explain how you determined the measures d and h. ⓗ



11. The Islamic design below right is based on the star decagon construction shown below left. The ten angles surrounding the center are all congruent. Find the lettered angle measures. How many triangles are not isosceles? ⓗ

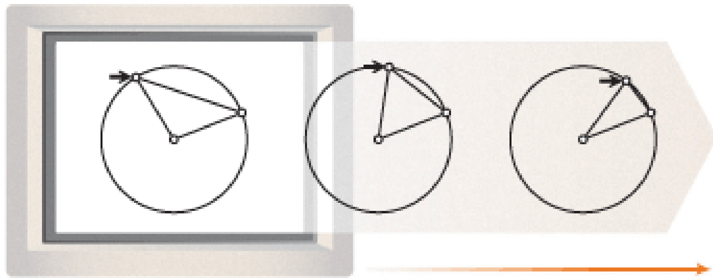




12. Study the triangles in the software constructions below. Each triangle has one vertex at the center of the circle, and two vertices on the circle.

a. Are the triangles all isosceles? Write a paragraph proof explaining why or why not.

b. If the vertex at the center of the first circle has an angle measure of 60°, find the measures of the other two angles in that triangle.



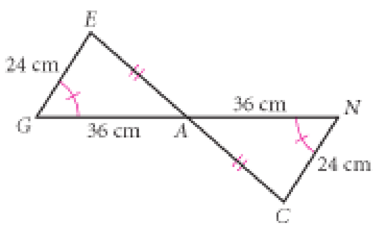
[For an interactive version of this sketch, see the **Dynamic Geometry Exploration** Triangles in a Circle at www.keymath.com/DG .]

keymath.com/DG

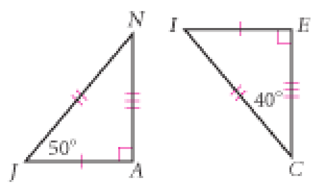
Review

In Exercises 13 and 14, complete the statement of congruence from the information given. Remember to write the statement so that corresponding parts are in order.

13. $\triangle GEA \cong \triangle$?

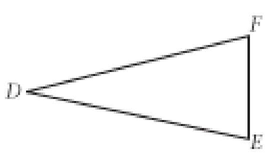


14. $\triangle JAN \cong \triangle$?

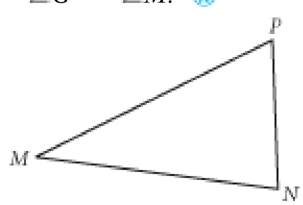


In Exercises 15 and 16, use a compass or patty paper, and a straightedge, to construct a triangle that is not congruent to the given triangle, but has the given parts congruent. The symbol means "not congruent to."

15. ***Construction*** Construct $\triangle ABC \cong \triangle DEF$ with $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$.



16. ***Construction*** Construct $\triangle GHK \cong \triangle MNP$ with $\overline{HK} \cong \overline{NP}$, $\overline{GH} \cong \overline{MN}$, and $\angle G \cong \angle M$.



17. ***Construction*** With a straightedge and patty paper, construct an angle that measures 105°.

In Exercises 18–21, determine whether each pair of lines through the points below is parallel, perpendicular, or neither.

$A(1, 3)$ $B(6, 0)$ $C(4, 3)$ $D(1, -2)$ $E(-3, 8)$ $F(-4, 1)$ $G(-1, 6)$ $H(4, -4)$

18. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$

19. $\overleftrightarrow{FG}$ and $\overleftrightarrow{CD}$

20. $\overleftrightarrow{AD}$ and $\overleftrightarrow{CH}$

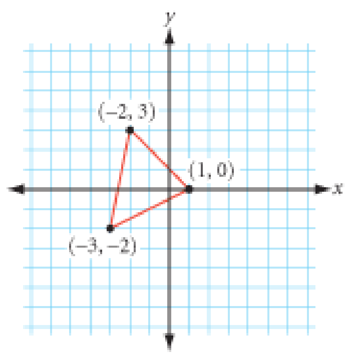
21. $\overleftrightarrow{DE}$ and $\overleftrightarrow{GH}$

22. Using the preceding coordinate points, is *FGCD* a trapezoid, a parallelogram, or neither?

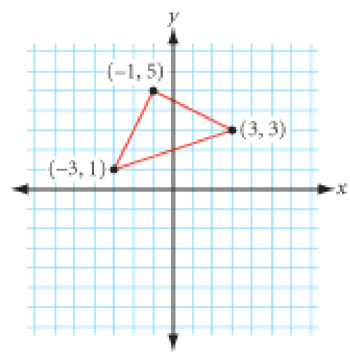
23. Picture the isosceles triangle below toppling side over side to the right along the line. Copy the triangle and line onto your paper, then construct the path of point *P* through two cycles. Where on the number line will the vertex point land?

For Exercises 24 and 25, use the ordered pair rule shown to relocate each of the vertices of the given triangle. Connect the three new points to create a new triangle. Is the new triangle congruent to the original one? Describe how the new triangle has changed position from the original.

24. $(x, y) \rightarrow (x + 5, y - 3)$ ⓗ



25. $(x, y) \rightarrow (x, -y)$



IMPROVING YOUR REASONING SKILLS

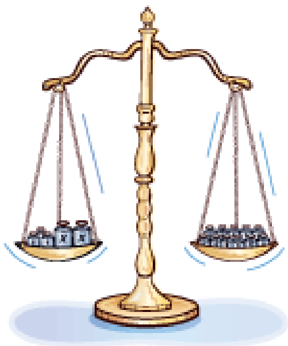
Hundreds Puzzle

Fill in the blanks of each equation below. All nine digits—1 through 9—must be used, in order! You may use any combination of signs for the four basic operations (+, −, ·, ÷), parentheses, decimal points, exponents, factorial signs, and square root symbols, and you may place the digits next to each other to create two-digit or three-digit numbers.

Example: 1 + 2(3 + 4.5) + 67 + 8 + 9 = 100

1. 1 + 2 + 3 − 4 + 5 + 6 + $\underline{?}$ + 9 = 100
2. 1 + 2 + 3 + 4 + 5 + $\underline{?}$ = 100
3. 1 + 2 + 3 . 4 . 5 / 6 + $\underline{?}$ = 100
4. (−1 − $\underline{?}$) / 5 + 6 + 7 + 89 = 100
5. 1 + 23 − 4 + $\underline{?}$ + 9 = 100

Solving Equations



In this chapter you may have already been solving some problems by setting up and solving an equation. An equation is a statement that two expressions are equal. The solution to an equation is the value (or values) of the variable that makes the equation true. The solution to the equation $2x + 3 = 11$ is $x = 4$. You can check this by substituting 4 for x to see that $2(4) + 3 = 11$ is a true equation.

While there are many properties of real numbers and properties of equality, here are some of the main properties that help you solve equations.

Distributive Property

$a(b + c) = a \cdot b + a \cdot c$

This property allows you to simplify equations by separating the terms within parentheses.

$3(5x - 7) = 15x - 21$

Combining like terms

$ax + bx = (a + b)x$

This process, based on the distributive property, allows you to simplify one side of an equation by adding the coefficients of expressions with the same variable.

$-4y + 9y = (-4 + 9)y = 5y$

Properties of Equality

Given $a = b$, for any number c,

Addition property

$a + c = b + c$

Subtraction property

$a - c = b - c$

Multiplication property

$ac = bc$

Division property

$\frac{a}{c} = \frac{b}{c}$ (provided $c \neq 0$)

These properties allow you to perform the same operation on both sides of an equation.

Substitution property

If $a = b$, then a can be replaced with b in any equation.

This property allows you to check your solution to an equation by replacing each variable with its value. Substitution is also used to solve some equations and in writing proofs.

EXAMPLE A

Solve $4x + 8 = -4(2x - 7) + 4$.

▸ Solution

$4x + 8 = -4(2x - 7) + 2x$	The original equation.
$4x + 8 = -8x + 28 + 2x$	Distribute.
$4x + 8 = -6x + 28$	Combine like terms.
$10x + 8 = 28$	Add $6x$ to both sides.
$10x = 20$	Subtract 8 from both sides.
$x = 2$	Divide both sides by 10.

The solution is $x = 2$.

Check that the solution makes the original equation true.

$4(2) + 8 \stackrel{?}{=} -4[2(2) - 7] + 2(2)$	Substitute 2 for x.
$4(2) + 8 \stackrel{?}{=} -4[-3] + 4$	Simplify following the order of operations.
$8 + 8 \stackrel{?}{=} 12 + 4$	
$16 = 16$	The solution checks.

When an equation contains fractions or rational expressions, it is sometimes easiest to "clear" them by multiplying both sides of the equation by a common denominator.

EXAMPLE B

Solve $\frac{x}{2} = \frac{3x}{5} - \frac{1}{4}$.

▸ Solution

The denominators are 2, 5, and 4. The least common denominator is 20.

$\frac{x}{2} = \frac{3x}{5} - \frac{1}{4}$	The original equation.
$20\left(\frac{x}{2}\right) = 20\left(\frac{3x}{5} - \frac{1}{4}\right)$	Multiply both sides by 20.
$\frac{20x}{2} = \frac{60x}{5} - \frac{20}{4}$	Distribute.
$10x = 12x - 5$	Reduce the fractions. Now solve as you did in Example A.
$2x = -5$	Subtract $12x$ from both sides.
$x = 2.5$	Divide both sides by -2.

Check the solution.

$\frac{2.5}{2} \stackrel{?}{=} \frac{3(2.5)}{5} - \frac{1}{4}$	Substitute 2.5 for x.
$1.25 \stackrel{?}{=} 1.5 - 0.25$	Simplify following the order of operations.
$1.25 = 1.25$	

EXERCISES

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In Exercises 1–2, state whether each equation is true or false.

1. $2(4+5) = 13$

2. $2 + [3(-4) - 4] = 2(-4 - 3)$

In Exercises 3–5, determine whether the value given for the variable is a solution to the equation.

3. $x - 8 = 2;\ x = 6$

4. $4(3y - 1) = -40;\ y = -3$

5. $\frac{3}{4}n - \frac{1}{2} = \frac{1}{8};\ n = \frac{1}{4}$

In Exercises 6–13, solve the equation and check your solution.

6. $6x - 3 = 39$

7. $3y - 7 = 5y + 1$

8. $6x - 4(3x + 8) = 16$

9. $7 - 3(2x - 5) = 1 - x$

10. $5(n - 2) - 14n = -3n - (5 - 4n)$

11. $\frac{1}{6} = \frac{3}{10} - \frac{1}{15}x$

12. $\frac{4}{t} - \frac{1}{6} = \frac{1}{t}$

13. $\frac{3n}{4} - \frac{1}{3} = \frac{2n - 1}{6}$

14. A **proportion** is a statement of equality between two ratios. For example, $\frac{3}{4} = \frac{x}{3}$ is a proportion.

a. Solve the proportion above by clearing the fractions, as in Example B.

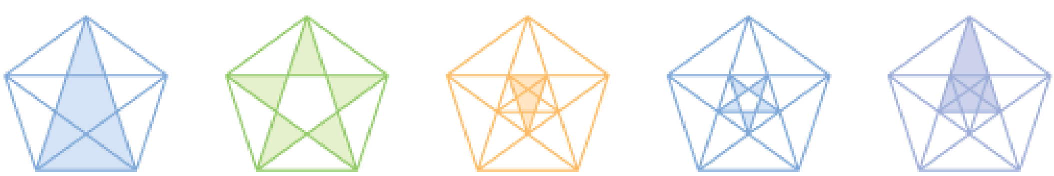
b. You may have previously learned that you can solve a proportion by "cross multiplying." If so, use this method to solve the proportion above, and compare this method to the one you used in part a.

15. Try solving $2(3x + 1) = 6x + 3$. Explain what happens. What is the solution?

16. Below is Camella's work solving an equation. Camella says, "The solution is zero." Is she correct? Explain.

$2x + 2(x - 1) = 4(x - 3) + 10$	Original equation.
$2x + 2x - 2 = 4x - 12 + 10$	Distribute.
$4x - 2 = 4x - 2$	Combine like terms.
$4x = 4x$	Add 2 to both sides.
$0 = 0$	Subtract $4x$ from both sides.

17. A golden triangle is an isosceles triangle with many special properties. One of them is the measure of either of its base angles is twice the measure of its vertex angle. Sketch a golden triangle with variables representing the angles. Apply previous conjectures to write an equation. Then solve it to determine the measure of each angle.



The diagonals of a regular pentagon form many different golden triangles.

Student Web Links @ Keymath.com

LESSON 4.3

Triangle Inequalities

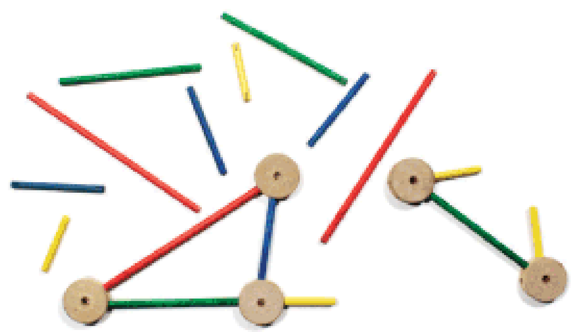
Readers are plentiful, thinkers are rare.
HARRIET MARTINEAU

How long must each side of this drawbridge be so that the bridge spans the river when both sides come down?

Drawbridges over the Chicago River in Chicago, Illinois

The sum of the lengths of the two parts of the drawbridge must be equal to or greater than the distance across the waterway. Triangles have similar requirements.

You can build a triangle using one blue rod, one green rod, and one red rod. Could you still build a triangle if you used a yellow rod instead of the green rod? Why or why not? Could you form a triangle with two yellow rods and one green rod? What if you used two green rods and one yellow rod?



How can you determine which sets of three rods can be arranged into triangles and which can't? How do the measures of the angles in the triangle relate to the lengths of the rods? How is measure of the exterior angle formed by the yellow and blue rods in the triangle above related to the measures of the angles inside the triangle? In this lesson you will investigate these questions.

Investigation 1

What Is the Shortest Path from *A* to *B*?

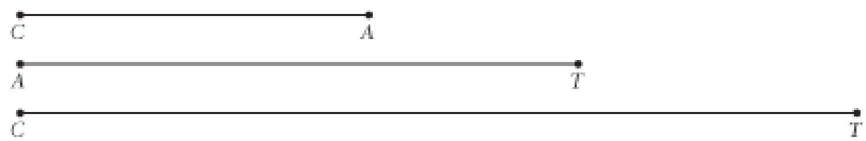
You will need

- a compass
- a straightedge

Each person in your group should do each construction. Compare results when you finish.

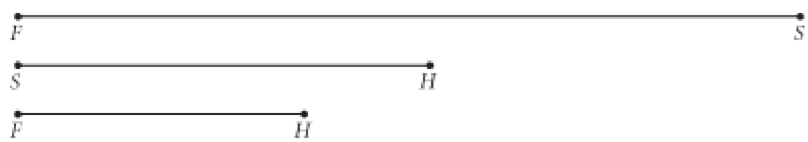
Step 1 Construct a triangle with each set of segments as sides.

Given:



Construct: $\triangle CAT$

Given:



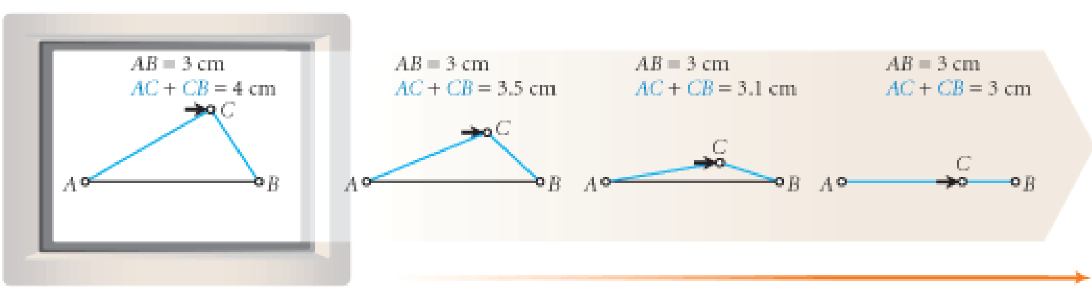
Construct: $\triangle FSH$

Step 2 You should have been able to construct $\triangle CAT$, but not $\triangle FSH$. Why? Discuss your results with others. State your observations as your next conjecture.

Triangle Inequality Conjecture C-20

The sum of the lengths of any two sides of a triangle is _?_ the length of the third side.

The Triangle Inequality Conjecture relates the lengths of the three sides of a triangle. You can also think of it in another way: The shortest path between two points is along the segment connecting them. In other words, the path from *A* to *C* to *B* can't be shorter than the path from *A* to *B*.



keymath.com/DG

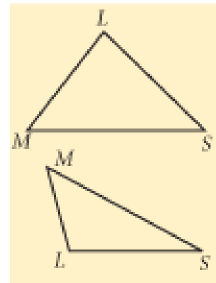
For an interactive version of this sketch, see the **Dynamic Geometry Exploration** The Triangle Inequality at www.keymath.com/DG

Investigation 2
Where Are the Largest and Smallest Angles?

You will need

- a ruler
- a protractor

Each person should draw a different scalene triangle for this investigation. Some group members should draw acute triangles, and some should draw obtuse triangles.



Step 1 Measure the angles in your triangle. Label the angle with greatest measure $\angle L$, the angle with second greatest measure $\angle M$, and the smallest angle $\angle S$.

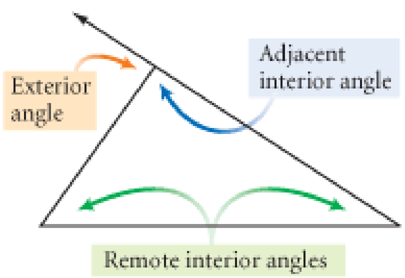
Step 2 Measure the three sides. Label the longest side l, the second longest side m, and the shortest side s.

Step 3 Which side is opposite $\angle L$? $\angle M$? $\angle S$?

Discuss your results with others. Write a conjecture that states where the largest and smallest angles are in a triangle, in relation to the longest and shortest sides.

Side-Angle Inequality Conjecture C-21

In a triangle, if one side is longer than another side, then the angle opposite the longer side is $\underline{\ ?\ }$.



So far in this chapter, you have studied interior angles of triangles. Triangles also have exterior angles. If you extend one side of a triangle beyond its vertex, then you have constructed an **exterior angle** at that vertex.

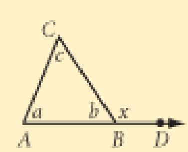
Each exterior angle of a triangle has an **adjacent interior angle** and a pair of **remote interior angles.** The remote interior angles are the two angles in the triangle that do not share a vertex with the exterior angle.

Investigation 3
Exterior Angles of a Triangle

You will need

- a straightedge
- patty paper

Each person should draw a different scalene triangle for this investigation. Some group members should draw acute triangles, and some should draw obtuse triangles.

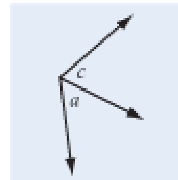


Step 1 On your paper, draw a scalene triangle, $\triangle ABC$. Extend $\overline{AB}$ beyond point B and label a point D outside the triangle on $\overrightarrow{AB}$. Label the angles as shown.

Step 2 Copy the two remote interior angles, $\angle A$ and $\angle C$, onto patty paper to show their sum.

Step 3 How does the sum of a and c compare with x? Use your patty paper from Step 2 to compare.

Step 4 Discuss your results with your group. State your observations as a conjecture.



Triangle Exterior Angle Conjecture C-22

The measure of an exterior angle of a triangle __?__ .

Developing Proof The investigation may have convinced you that the Triangle Exterior Angle Conjecture is true, but can you explain *why* it is true for every triangle?

As a group, discuss how to prove the Triangle Exterior Angle Conjecture. Use reasoning strategies such as draw a labeled diagram, represent a situation algebraically, and apply previous conjectures. Start by making a diagram and listing the relationships you already know among the angles in the diagram, then plan out the logic of your proof.

You will write the paragraph proof of the Triangle Exterior Angle Conjecture in Exercise 17. ■

Keymath.com Practice Your Skills

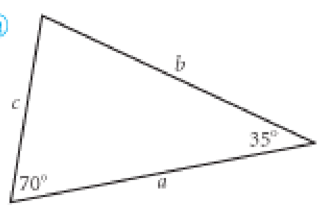
EXERCISES

In Exercises 1–4, determine whether it is possible to draw a triangle with sides having the given measures. If possible, write yes. If not possible, write no and make a sketch demonstrating why it is not possible.

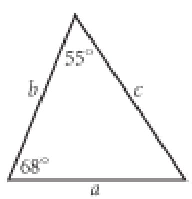
1. 3 cm, 4 cm, 5 cm **2.** 4 m, 5 m, 9 m **3.** 5 ft, 6 ft, 12 ft **4.** 3.5 cm, 4.5 cm, 7 cm

In Exercises 5–10, use your new conjectures to arrange the unknown measures in order from greatest to least.

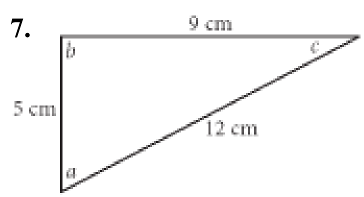
5. (h)



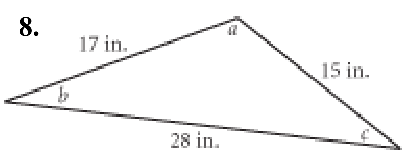
6.



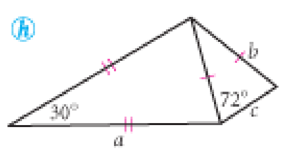
7.



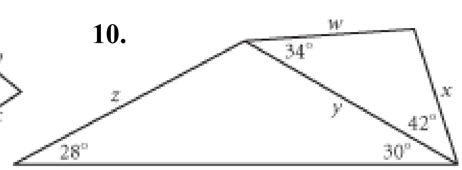
8.



9. (h)



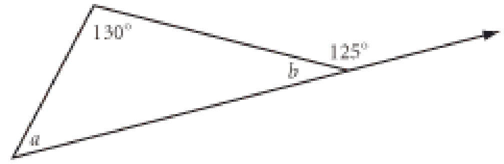
10.



11. If 54 and 48 are the lengths of two sides of a triangle, what is the range of possible values for the length of the third side? (h)

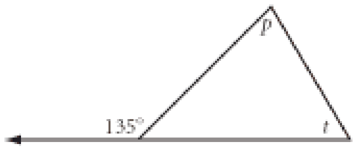
12. ***Developing Proof*** What's wrong with this picture? Explain.

13. ***Developing Proof*** What's wrong with this picture? Explain.

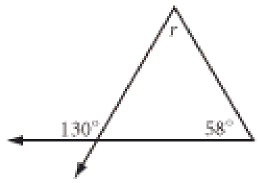


In Exercises 14–16, use one of your new conjectures to find the missing measures.

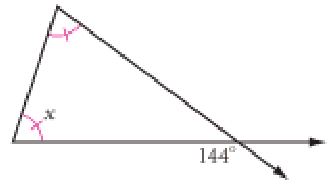
14. $t + p = \underline{?}$



15. $r = \underline{?}$

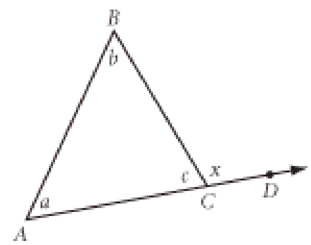


16. $x = \underline{?}$



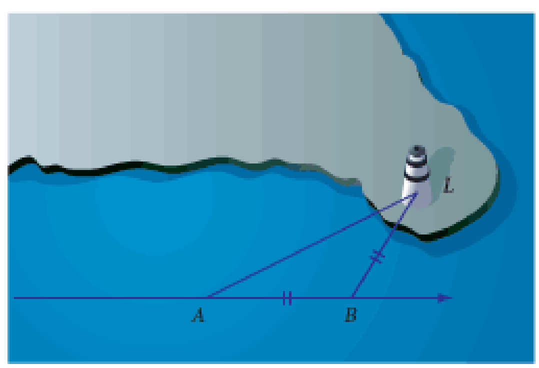
17. ***Developing Proof*** Use the Triangle Sum Conjecture to explain why the Triangle Exterior Angle Conjecture is true. Use the figure at right.

18. Read the Recreation Connection below. If you want to know the perpendicular distance from a landmark to the path of your boat, what should be the measurement of your bow angle when you begin recording?



Recreation CONNECTION

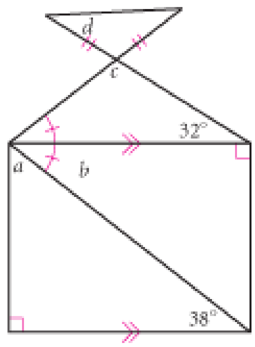
Geometry is used quite often in sailing. For example, to find the distance between the boat and a landmark on shore, sailors use a rule called *doubling the angle on the bow.* The rule says, measure the angle on the bow (the angle formed by your path and your line of sight to the landmark, also called your bearing) at point *A*. Check your bearing until, at point *B,* the bearing is double the reading at point *A*. The distance traveled from *A* to *B* is also the distance from the landmark to your new position.



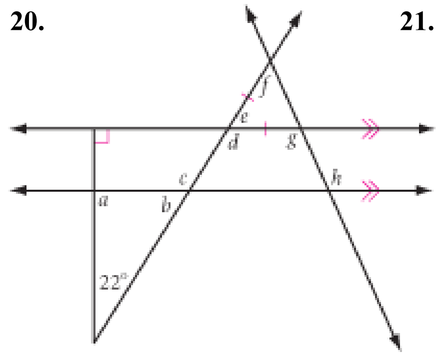
Review

In Exercises 19 and 20, calculate each lettered angle measure.

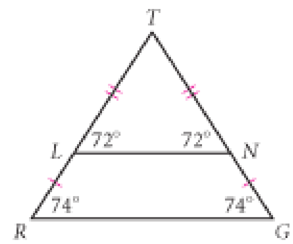
19.



20.

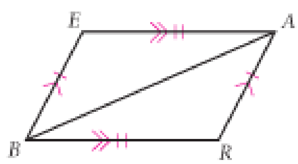


21. What's wrong with this picture of $\triangle TRG$? Explain.

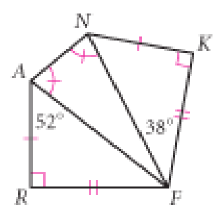


In Exercises 22–24, complete the statement of congruence.

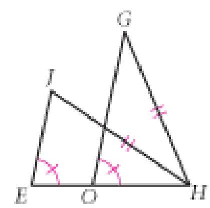
22. $\triangle BAR \cong \triangle$?



23. $\triangle FAR \cong \triangle$?



24. $\overline{HG} \cong \overline{HJ}$
$\triangle HEJ \cong \triangle$?



project

RANDOM TRIANGLES

Imagine you cut a 20cm straw in two randomly selected places anywhere along its length. What is the probability that the three pieces will form a triangle? How do the locations of the cuts affect whether or not the pieces will form a triangle? Explore this situation by cutting a straw in different ways, or use geometry software to model different possibilities. Based on your informal exploration, predict the probability of the pieces forming a triangle.

Now generate a large number of randomly chosen lengths to simulate the cutting of the straw. Analyze the results and calculate the probability based on your data. How close was your prediction?

Your project should include

- Your prediction and an explanation of how you arrived at it.
- Your randomly generated data.
- An analysis of the results and your calculated probability.
- An explanation of how the location of the cuts affects the chances of a triangle being formed.

Fathom 2
Dynamic Data Software

You can use Fathom to generate many sets of random numbers quickly. You can also set up tables to view your data, and enter formulas to calculate quantities based on your data.

LESSON

4.4

Are There Congruence Shortcuts?

The person who knows how will always have a job; the person who knows why will always be that person's boss.

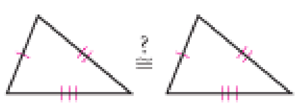
ANONYMOUS

A building contractor has just assembled two massive triangular trusses to support the roof of a recreation hall. Before the crane hoists them into place, the contractor needs to verify that the two triangular trusses are identical. Must the contractor measure and compare all six parts of both triangles?

You learned from the Third Angle Conjecture that if there is a pair of angles congruent in each of two triangles, then the third angles must be congruent. But will this guarantee that the trusses are the same size? You probably need to also know something about the sides in order to be sure that two triangles are congruent. Recall from earlier exercises that *fewer* than three parts of one triangle can be congruent to corresponding parts of another triangle, without the triangles being congruent.

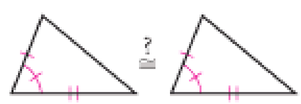
So let's begin looking for congruence shortcuts by comparing three parts of each triangle. There are six different ways that the three corresponding parts of two triangles may be congruent. They are diagrammed below. Some of these will be congruence shortcuts, and some will not.

Side-Side-Side (SSS)



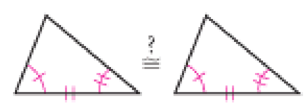
Three pairs of congruent sides

Side-Angle-Side (SAS)



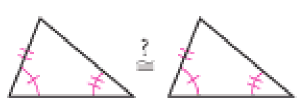
Two pairs of congruent sides and one pair of congruent angles (angles between the pairs of sides)

Angle-Side-Angle (ASA)



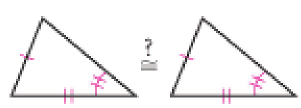
Two pairs of congruent angles and one pair of congruent sides (sides between the pairs of angles)

Side-Angle-Angle (SAA)



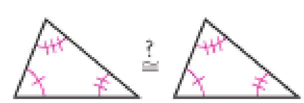
Two pairs of congruent angles and one pair of congruent sides (sides not between the pairs of angles)

Side-Side-Angle (SSA)



Two pairs of congruent sides and one pair of congruent angles (angles not between the pairs of sides)

Angle-Angle-Angle (AAA)



Three pairs of congruent angles

You will investigate three of these cases in this lesson and the other three in the next lesson to discover which of these six possible cases turn out to be congruence shortcuts and which do not.

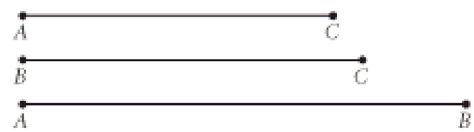
Investigation 1
Is SSS a Congruence Shortcut?

You will need

- a compass or patty paper
- a straightedge

First you will investigate the Side-Side-Side (SSS) case. If the three sides of one triangle are congruent to the three sides of another, must the two triangles be congruent?

Step 1 Construct a triangle from the three parts shown. Be sure you match up the endpoints labeled with the same letter. If you need help with this construction, see page 170, Example A.



Step 2 Compare your triangle with the triangles made by others in your group. (One way to compare them is to place the triangles on top of each other and see if they coincide.) Is it possible to construct different triangles from the same three parts, or will all the triangles be congruent?

Step 3 You are now ready to complete the conjecture for the SSS case.

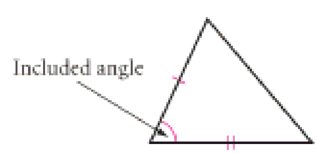
SSS Congruence Conjecture C-23

If the three sides of one triangle are congruent to the three sides of another triangle, then _?_.

Career CONNECTION

Congruence is very important in design and manufacturing. Modern assembly-line production relies on identical, or congruent, parts that are interchangeable. In the assembly of an automobile, for example, the same part needs to fit into each car coming down the assembly line.

An angle that is included between two sides of a triangle is called an **included angle,** as shown in the diagram at right. You will investigate this case next.

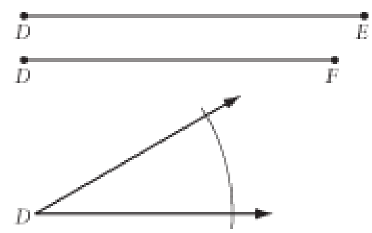


Investigation 2
Is SAS a Congruence Shortcut?

You will need
- a compass or patty paper
- a straightedge

Next you will consider the Side-Angle-Side (SAS) case. If two sides and the included angle of one triangle are congruent to two sides and the included angle of another, must the triangles be congruent?

Step 1 Construct a triangle from the three parts shown. Be sure you match up the endpoints labeled with the same letter. If you need help with this construction, see page 171, Exercise 2.



Step 2 Compare your triangle with the triangles made by others in your group. Is it possible to construct different triangles from the same three parts, or will all the triangles be congruent?

Step 3 You are now ready to complete the conjecture for the SAS case.

SAS Congruence Conjecture C-24

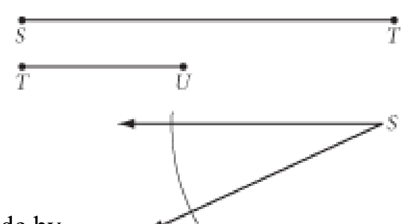
If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then __?__.

Investigation 3
Is SSA a Congruence Shortcut?

You will need
- a compass or patty paper
- a straightedge

Finally you will consider the Side-Side-Angle (SSA) case. If two sides and a non-included angle of one triangle are congruent to the corresponding two sides and non-included angle of another, must the triangles be congruent?

Step 1 Construct a triangle from the three parts shown. Be sure you match up the endpoints labeled with the same letter. If you need help with this construction, see page 172, Exercise 5.



Step 2 Compare your triangle with the triangles made by others in your group. Is it possible to construct different triangles from the same three parts, or will all the triangles be congruent?

Step 3 If two sides and a non-included angle of one triangle are congruent to the corresponding two sides and non-included angle of another triangle, do the two triangles have to be congruent? Explain why or show a counterexample.

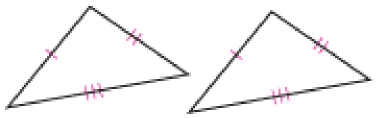
EXERCISES

You will need

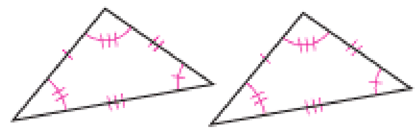
Construction tools for Exercises 21 and 22

1. The picture statement below represents the SSS Triangle Congruence Conjecture. Explain what the picture statement means.

If you know this:



then you also know this:

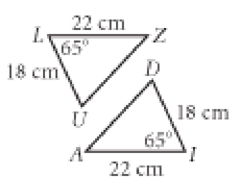


2. Create a picture statement to represent the SAS Triangle Congruence Conjecture. Explain what the picture statement means.

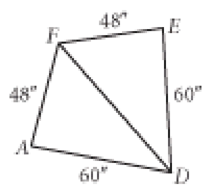
3. In the third investigation you discovered that the SSA case is not a triangle congruence shortcut. Sketch a counterexample to show why.

For Exercises 4–9, decide whether the triangles are congruent, and name the congruence shortcut you used. If the triangles cannot be shown to be congruent as labeled, write "cannot be determined."

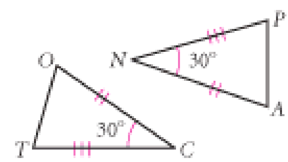
4. Which conjecture tells you $\triangle LUZ \cong \triangle IDA$? ⓗ



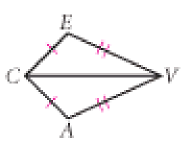
5. Which conjecture tells you $\triangle AFD \cong \triangle EFD$? ⓗ



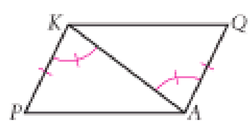
6. Which conjecture tells you $\triangle COT \cong \triangle NPA$?



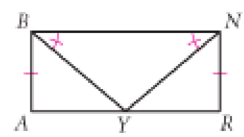
7. Which conjecture tells you $\triangle CAV \cong \triangle CEV$?



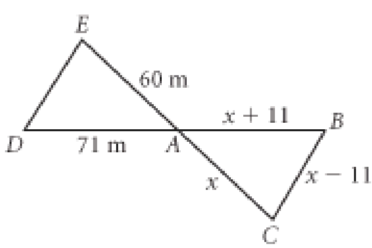
8. Which conjecture tells you $\triangle KAP \cong \triangle AKQ$?



9. Y is a midpoint. Which conjecture tells you $\triangle AYB \cong \triangle RYN$?



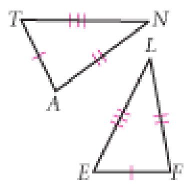
10. The perimeter of $\triangle ABC$ is 180 m. Is $\triangle ABC \cong \triangle ADE$? Which conjecture supports your conclusion?



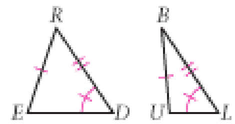
11. Explain why the boards that are nailed diagonally in the corners of this wooden gate make the gate stronger and prevent it from changing its shape under stress.

Developing Proof In Exercises 12–17, if possible, name a triangle congruent to the given triangle and state the congruence conjecture. If you cannot show any triangles to be congruent from the information given, write "cannot be determined" and explain why.

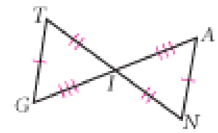
12. $\triangle ANT \cong \triangle \underline{?}$



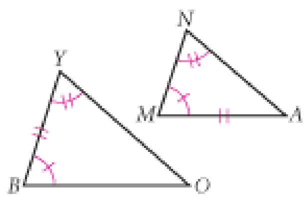
13. $\triangle RED \cong \triangle \underline{?}$



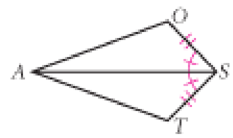
14. $\triangle GIT \cong \triangle \underline{?}$



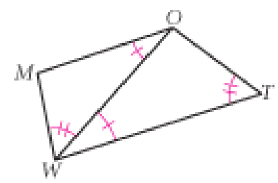
15. $\triangle MAN \cong \triangle \underline{?}$



16. $\triangle SAT \cong \triangle \underline{?}$

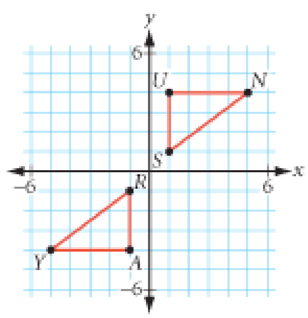


17. $\triangle WOM \cong \triangle \underline{?}$

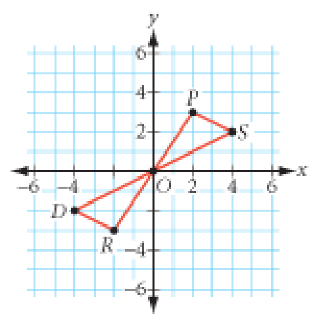


In Exercises 18 and 19, determine whether the segments or triangles in the coordinate plane are congruent and explain your reasoning.

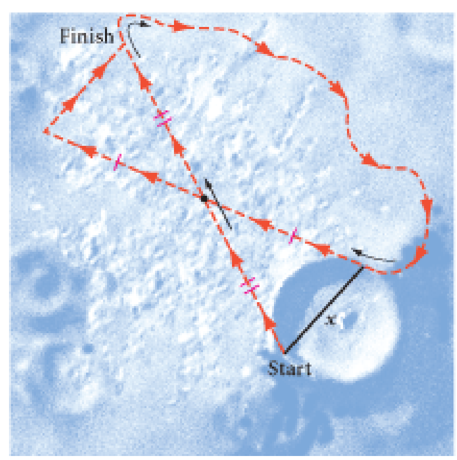
18. $\triangle SUN \cong \triangle \underline{?}$



19. $\triangle DRO \cong \triangle \underline{?}$



20. NASA scientists using a lunar exploration vehicle (LEV) wish to determine the distance across the deep crater shown at right. They have mapped out a path for the LEV as shown. What do the scientists need to measure to find the approximate distance across the crater? Explain why.



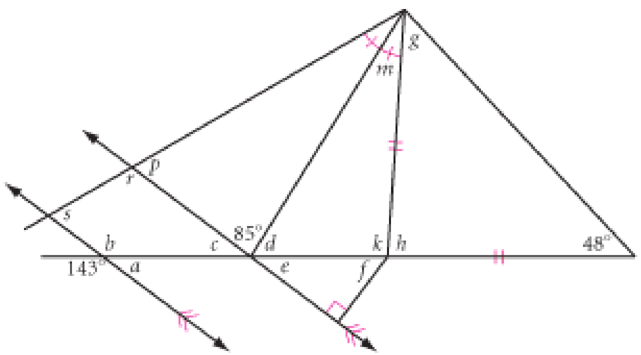
In Exercises 21 and 22, use a compass and straightedge, or patty paper, to perform these constructions.

21. ***Construction*** Draw a triangle. Use the SSS Congruence Conjecture to construct a second triangle congruent to the first.

22. ***Construction*** Draw a triangle. Use the SAS Congruence Conjecture to construct a second triangle congruent to the first.

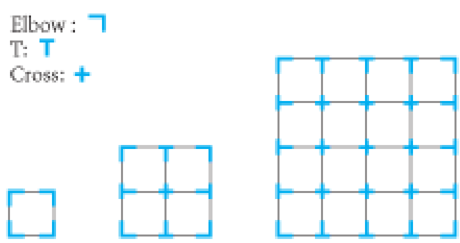
Review

23. ***Developing Proof*** Copy the figure. Calculate the measure of each lettered angle. Explain how you determined measures h and s.



24. If two sides of a triangle measure 8 cm and 11 cm, what is the range of values for the length of the third side?

25. How many "elbow," "T," and "cross" pieces do you need to build a 20-by-20 grid? Start with the smaller grids shown below. Copy and complete the table.



Side length	1	2	3	4	5	...	n	...	20
Elbows	4	4							
T's	0	4							
Crosses	0	1							

26. Solve each equation for y.

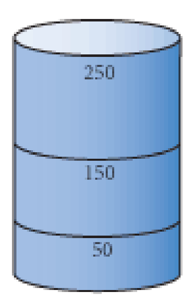
a. $2y - 5(8 - y) = 2$ **b.** $\frac{y}{2} - \frac{1}{3} = \frac{y+3}{4}$ **c.** $3x + 4y = 8$

27. Isosceles right triangle ABC has vertices with coordinates $A(-8, 2)$, $B(-5, -3)$, and $C(0, 0)$. Find the coordinates of the orthocenter.

IMPROVING YOUR REASONING SKILLS

Container Problem I

You have a small cylindrical measuring glass with a maximum capacity of 250 mL. All the marks have worn off except the 150 mL and 50 mL marks. You also have a large unmarked container. It is possible to fill the large container with exactly 350 mL. How? What is the fewest number of steps required to obtain 350 mL?



LESSON

4.5

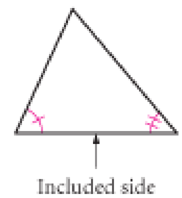
Are There Other Congruence Shortcuts?

There is no more a math mind, than there is a history or an English mind.

GLORIA STEINEM

In the last lesson, you discovered that there are six ways that three parts of two triangles can be the same. You found that SSS and SAS both lead to the congruence of the two triangles, but that SSA does not. In this lesson you will investigate the other three cases.

A side that is included between two angles of a triangle is called an **included side,** as shown in the diagram at right. You will investigate this case next.



Investigation 1

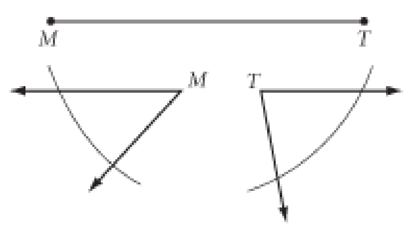
Is ASA a Congruence Shortcut?

You will need

- a compass or patty paper
- a straightedge

First you will consider the Angle-Side-Angle (ASA) case. If two angles and the included side of one triangle are congruent to two angles and the included side of another, must the triangles be congruent?

Step 1 Construct a triangle from the three parts shown. Be sure you match up the angles with the endpoints labeled with the same letter. If you need help with this construction, see page 171, Exercise 3.



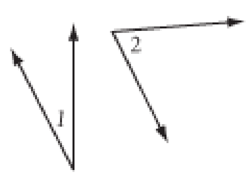
Step 2 Compare your triangle with the triangles made by others in your group. Is it possible to construct different triangles from the same three parts, or will all the triangles be congruent?

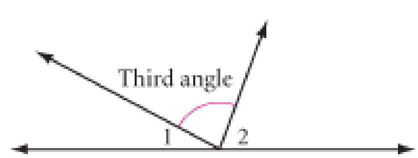
Step 3 You are now ready to complete the conjecture for the ASA case.

ASA Congruence Conjecture C-25

If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then $\underline{?}$.

In Lesson 4.1, you learned that if you are given two angles of a triangle, you can construct the third angle using the Triangle Sum Conjecture.





Investigation 2

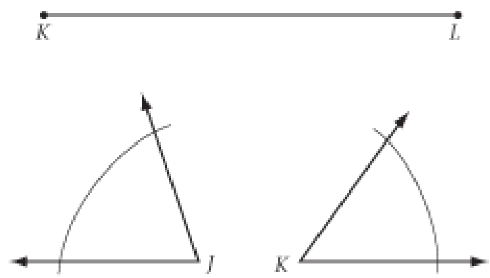
Is SAA a Congruence Shortcut?

You will need

- a compass or patty paper
- a straightedge

Next you will consider the Side-Angle-Angle (SAA) case. If two angles and a non-included side of one triangle are congruent to the corresponding two angles and non-included side of another, must the triangles be congruent?

Step 1 Construct a triangle from the three parts shown. Be sure you match up the angles with the endpoints labeled with the same letter. If you need help with this construction, see page 204, Exercise 12.



Step 2 Compare your triangle with the triangles made by others in your group. Is it possible to construct different triangles from the same three parts, or will all the triangles be congruent?

Step 3 You are now ready to complete the conjecture for the SAA case.

SAA Congruence Conjecture C-26

If two angles and a non-included side of one triangle are congruent to the corresponding two angles and non-included side of another triangle, then $\underline{?}$.

Investigation 3

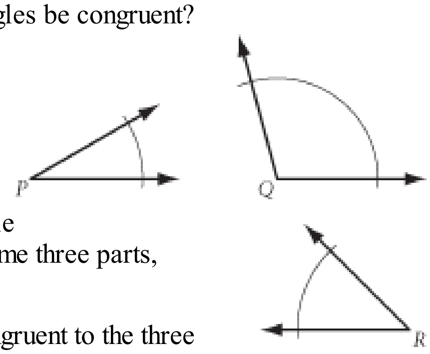
Is AAA a Congruence Shortcut?

You will need

- a compass or patty paper
- a straightedge

Finally you will investigate the Angle-Angle-Angle (AAA) case. If the three angles of one triangle are congruent to the three angles of another, must the triangles be congruent?

Step 1 Construct a triangle from the three parts shown. If you need help with this construction, see page 170, Example B.



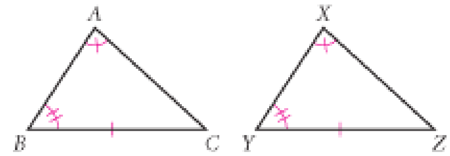
Step 2 Compare your triangle with the triangles made by others in your group. Is it possible to construct different triangles from the same three parts, or will all the triangles be congruent?

Step 3 If the three angles of one triangle are congruent to the three angles of another triangle, do the two triangles have to be congruent? Explain why or show a counterexample.

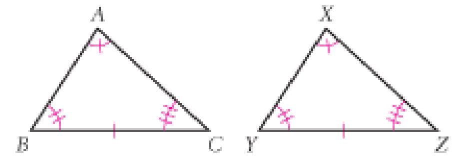
In Investigation 2 you found the SAA Congruence Conjecture inductively. You can also derive it deductively from the ASA Congruence Conjecture.

EXAMPLE

In triangles ABC and XYZ, $\angle A \cong \angle X$, $\angle B \cong \angle Y$, and $BC \cong YZ$. Is $\triangle ABC \cong \triangle XYZ$? Explain your answer in a paragraph.



► Solution



Two angles in one triangle are congruent to two angles in another. The Third Angle Conjecture says that $C \cong Z$. So you now have two angles and the *included* side of one triangle congruent to two angles and the included side of another. By the ASA Congruence Conjecture, $\triangle ABC \cong \triangle XYZ$.

Keymath.com Practice Your Skills

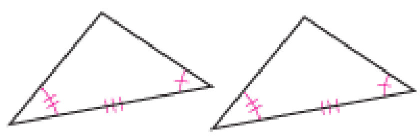
EXERCISES

You will need

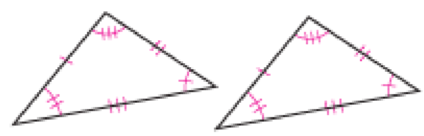
Construction tools for Exercises 20–23 and 26

1. The picture statement below represents the ASA Triangle Congruence Conjecture. Explain what the picture statement means.

If you know this:



then you also know this:

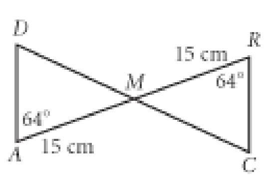


2. Create a picture statement to represent the SAA Triangle Congruence Conjecture. Explain what the picture statement means.

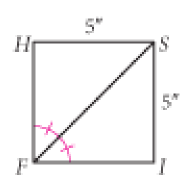
3. In the third investigation you discovered that the AAA case is not a triangle congruence shortcut. Sketch a counterexample to show why.

For Exercises 4–9, determine whether the triangles are congruent, and name the congruence shortcut. If the triangles cannot be shown to be congruent, write "cannot be determined."

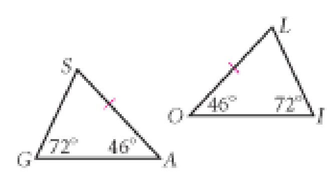
4. $\triangle AMD \cong \triangle RMC$



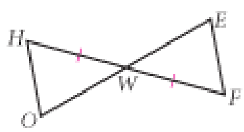
5. $\triangle FSH \cong \triangle FSI$



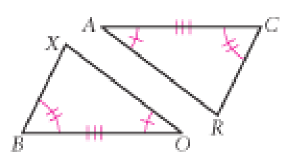
6. $\triangle GAS \cong \triangle IOL$ ⓗ



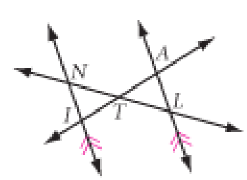
7. $\triangle HOW \cong \triangle FEW$



8. $\triangle BOX \cong \triangle CAR$

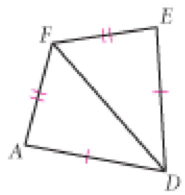


9. $\triangle ALT \cong \triangle INT$

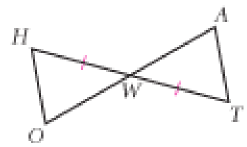


Developing Proof In Exercises 10–17, if possible, name a triangle congruent to the triangle given and state the congruence conjecture. If you cannot show any triangles to be congruent from the information given, write "cannot be determined" and explain why.

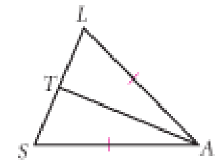
10. $\triangle FAD \cong \triangle \underline{?}$



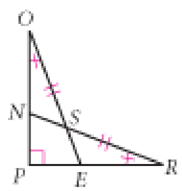
11. $\overline{OH} \parallel \overline{AT}$
$\triangle WHO \cong \triangle \underline{?}$



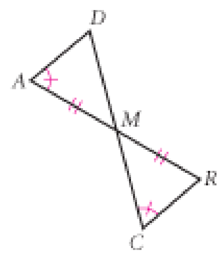
12. $\overline{AT}$ is an angle bisector.
$\triangle LAT \cong \triangle \underline{?}$



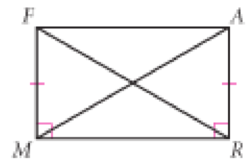
13. $PO = PR$
$\triangle POE \cong \triangle \underline{?}$
$\triangle SON \cong \triangle \underline{?}$



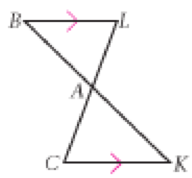
14. $\triangle \underline{?} \cong \triangle \underline{?}$



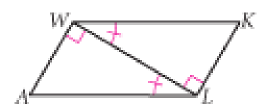
15. $\triangle RMF \cong \triangle \underline{?}$



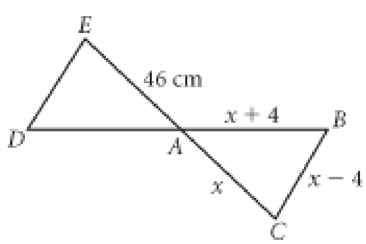
16. $\triangle BLA \cong \triangle \underline{?}$ ⓗ



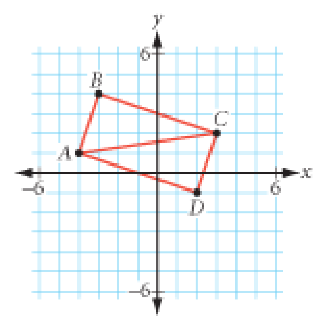
17. $\triangle LAW \cong \triangle \underline{?}$



18. The perimeter of $\triangle ABC$ is 138 cm and $\overline{BC} \parallel \overline{DE}$. Is $\triangle ABC \cong \triangle ADE$? Which conjecture supports your conclusion?



19. Use slope properties to show $\overline{AB} \perp \overline{BC}$, $\overline{CD} \perp \overline{DA}$, and $\overline{BC} \parallel \overline{DA}$. $\triangle ABC \cong \triangle \underline{?}$. Why?



In Exercises 20–22, use a compass or patty paper, and a straightedge, to perform each construction.

20. ***Construction*** Draw a triangle. Use the ASA Congruence Conjecture to construct a second triangle congruent to the first. Write a paragraph to justify your steps.

21. ***Construction*** Draw a triangle. Use the SAA Congruence Conjecture to construct a second triangle congruent to the first. Write a paragraph to justify your method.

22. ***Construction*** Construct two triangles that are not congruent, even though the three angles of one triangle are congruent to the three angles of the other. ⓗ

Review

23. ***Construction*** Using only a compass and a straightedge, construct an isosceles triangle with a vertex angle that measures 135°.

24. If n concurrent lines divide the plane into 250 parts then $n =$ _?_.

25. "If the two diagonals of a quadrilateral are perpendicular, then the quadrilateral is a rhombus." Explain why this statement is true or sketch a counterexample.

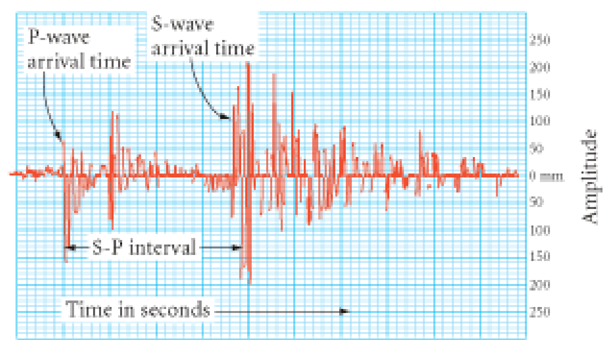
26. ***Construction*** Construct an isosceles right triangle with $\overline{KM}$ as one of the legs. How many noncongruent triangles can you construct? Why?

27. Sketch five lines in a plane that intersect in exactly five points. Now do this in a different way.

28. ***Application*** Scientists use seismograms and triangulation to pinpoint the epicenter of an earthquake.

a. Data recorded for one quake show that the epicenter is 480 km from Eureka, California; 720 km from Elko, Nevada; and 640 km from Las Vegas, Nevada. Trace the locations of these three towns and use the scale and your construction tools to find the location of the epicenter.

b. Is it necessary to have seismogram information from three towns? Would two towns suffice? Explain.



IMPROVING YOUR ALGEBRA SKILLS

Algebraic Sequences I

Find the next two terms of each algebraic sequence.

$x + 3y,\ 2x + y,\ 3x + 4y,\ 5x + 5y,\ 8x + 9y,\ 13x + 14y$, _?_, _?_

$x + 7y,\ 2x + 2y,\ 4x - 3y,\ 8x - 8y,\ 16x - 13y,\ 32x - 18y$, _?_, _?_

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LESSON

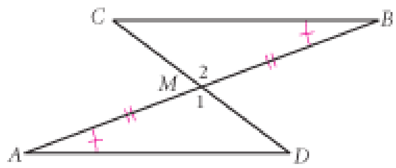
4.6

Corresponding Parts of Congruent Triangles

The job of the younger generation is to find solutions to the solutions found by the older generation.

ANONYMOUS

In Lessons 4.4 and 4.5, you discovered four shortcuts for showing that two triangles are congruent—SSS, SAS, ASA, and SAA. The definition of congruent triangles states that if two triangles are congruent, then the *corresponding* ***p****arts of those* ***c****ongruent* ***t****riangles are* ***c****ongruent*. We'll use the letters **CPCTC** to refer to the definition. Let's see how you can use congruent triangles and CPCTC.



EXAMPLE A

Is $\overline{AD} \cong \overline{BC}$ in the figure above? Use a deductive argument to explain why they must be congruent.

Solution

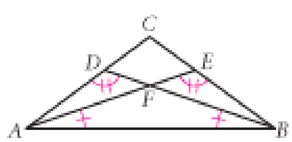
Here is one possible explanation: $\angle 1 \cong \angle 2$ because they are vertical angles. And it is given that $\overline{AM} \cong \overline{BM}$ and $\angle A \cong \angle B$. So, by ASA, $\triangle AMD \cong \triangle BMC$. Because the triangles are congruent, $\overline{AD} \cong \overline{BC}$ by CPCTC.

If you use a congruence shortcut to show that two triangles are congruent, then you can use CPCTC to show that any of their corresponding parts are congruent.

When you are trying to prove that triangles are congruent, it can be hard to keep track of what you know. Mark all the information on the figure. If the triangles are hard to see, you can use different colors or redraw them separately.

EXAMPLE B

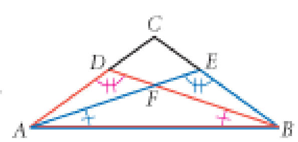
Is $\overline{AE} \cong \overline{BD}$? Write a paragraph proof explaining why.



Solution

The triangles you can use to show congruence are $\triangle ABD$ and $\triangle BAE$. You can separate or color them to see them more clearly.

Separated triangles



Color-coded triangles

You can see that the two triangles have two pairs of congruent angles and they share a side.

Paragraph Proof: Show that $\overline{AE} \cong \overline{BD}$.

In $\triangle ABD$ and $\triangle BAE$, $\angle D \cong \angle E$ and $\angle B \cong \angle A$. Also, $\overline{AB} \cong \overline{BA}$ because they are the same segment. So $\triangle ABD \cong \triangle BAE$ by SAA. By CPCTC, $\overline{AE} \cong \overline{BD}$. ■

Keymath.com Practice Your Skills

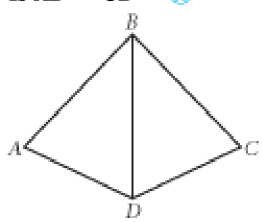
EXERCISES

You will need

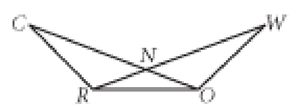
Construction tools for Exercises 16 and 17

▶ ***Developing Proof*** For Exercises 1–9, copy the figures onto your paper and mark them with the given information. Answer the question about segment or angle congruence. If your answer is yes, write a paragraph proof explaining why. Remember to use your reasoning strategies, especially apply previous conjectures and add an auxiliary line. If there is not enough information to prove congruence, write "cannot be determined," otherwise state which congruence shortcut you used.

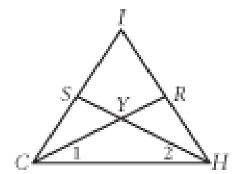
1. $\angle A \cong \angle C$, $\angle ABD \cong \angle CBD$
Is $\overline{AB} \cong \overline{CB}$? (h)



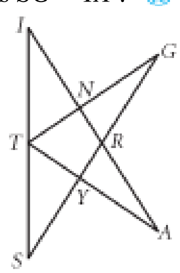
2. $\overline{CN} \cong \overline{WN}$, $\angle C \cong \angle W$
Is $\overline{RN} \cong \overline{ON}$? (h)



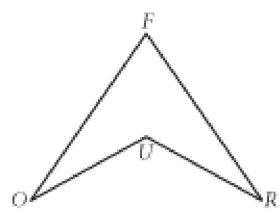
3. $\overline{CS} \cong \overline{HR}$, $\angle 1 \cong \angle 2$
Is $\overline{CR} \cong \overline{HS}$?



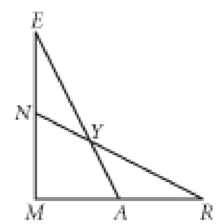
4. $\angle S \cong \angle I$, $\angle G \cong \angle A$
T is the midpoint of $\overline{SI}$.
Is $\overline{SG} \cong \overline{IA}$? (h)



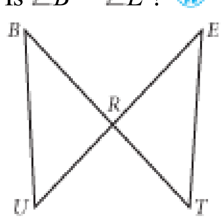
5. $\overline{FO} \cong \overline{FR}$, $\overline{UO} \cong \overline{UR}$
Is $\angle O \cong \angle R$? (h)



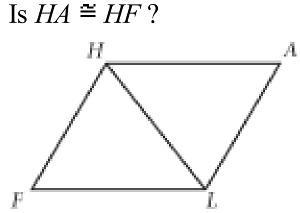
6. $\overline{MN} \cong \overline{MA}$, $\overline{ME} \cong \overline{MR}$
Is $\angle E \cong \angle R$?



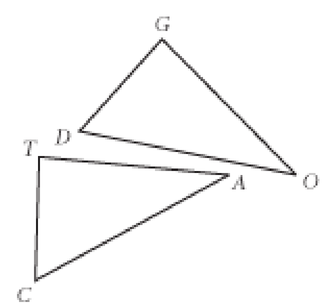
7. $\overline{BT} \cong \overline{EU}$, $\overline{BU} \cong \overline{ET}$
Is $\angle B \cong \angle E$? (h)



8. $HALF$ is a parallelogram.
Is $\overline{HA} \cong \overline{HF}$?

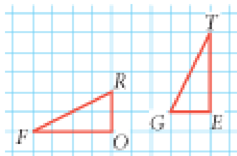


9. $\angle D \cong \angle C$, $\angle O \cong \angle A$,
$\angle G \cong \angle T$. Is $\overline{TA} \cong \overline{GO}$?

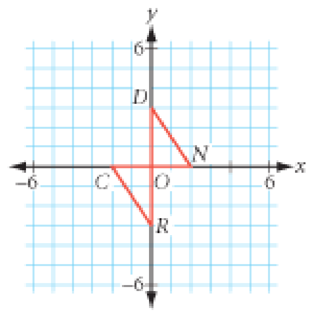


For Exercises 10 and 11, you can use the right angles and the lengths of horizontal and vertical segments shown on the grid. Answer the question about segment or angle congruence. If your answer is yes, explain why.

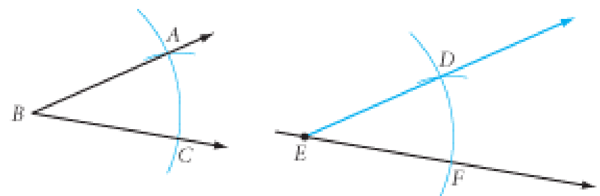
10. Is $\overline{FR} \cong \overline{GT}$? Why? ⓗ



11. Is $\angle OND \cong \angle OCR$? Why?



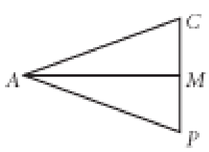
12. In Chapter 3, you used inductive reasoning to discover how to duplicate an angle using a compass and straightedge. Now you have the skills to explain *why* the construction works using deductive reasoning. The construction is shown at right. Write a paragraph proof explaining why it works.



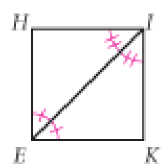
Review

In Exercises 13–15, complete each statement. If the figure does not give you enough information to show that the triangles are congruent, write "cannot be determined."

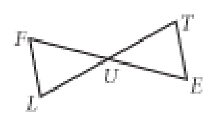
13. $\overline{AM}$ is a median.
$\triangle CAM \cong \triangle$?



14. $\triangle HEI \cong \triangle$?
Why?



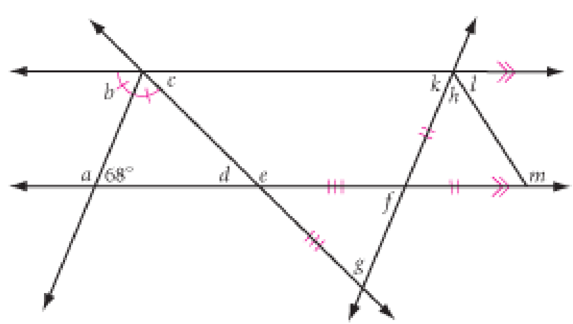
15. U is the midpoint of both $\overline{FE}$ and $\overline{LT}$. $\triangle ULF \cong \triangle$?



16. ***Construction*** Draw a triangle. Use the SAS Congruence Conjecture to construct a second triangle congruent to the first.

17. ***Construction*** Construct two triangles that are *not* congruent, even though two sides and a non-included angle of one triangle are congruent to two sides and a corresponding non-included angle of the other triangle. ⓗ

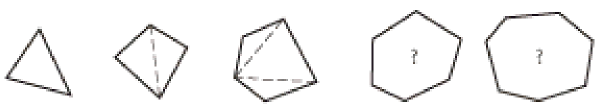
18. ***Developing Proof*** Copy the figure. Calculate the measure of each lettered angle. Explain how you determined the measures f and m.



19. According to math legend, the Greek mathematician Thales (ca. 625–547 B.C.E.) could tell how far out to sea a ship was by using congruent triangles. First, he marked off a long segment in the sand. Then, from each endpoint of the segment, he drew the angle to the ship. He then remeasured the two angles on the other side of the segment away from the shore. The point where the rays of these two angles crossed located the ship. What congruence conjecture was Thales using? Explain.

20. Isosceles right triangle ABC has vertices $A(-8, 2)$, $B(-5, -3)$, and $C(0, 0)$. Find the coordinates of the circumcenter.

21. The SSS Congruence Conjecture explains why triangles are rigid structures though other polygons are not. By adding one "strut" (diagonal) to a quadrilateral you create a quadrilateral that consists of two triangles, and that makes it rigid. What is the minimum number of struts needed to make a pentagon rigid? A hexagon? A dodecagon? What is the minimum number of struts needed to make other polygons rigid? Complete the table and make your conjecture.



Number of sides	3	4	5	6	7	...	12	...	n
Number of struts needed to make polygon rigid						...		...	

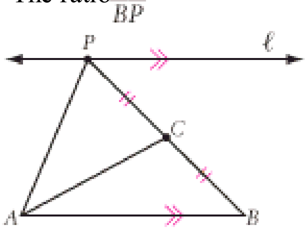
22. Line ℓ is parallel to $\overline{AB}$ If P moves to the right along ℓ, which of the following always decreases?

A. The distance PC

B. The distance from C to $\overline{AB}$

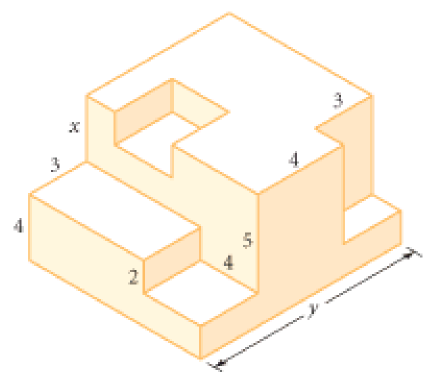
C. The ratio $\frac{AB}{AP}$

D. The ratio $\frac{BC}{BP}$



For an interactive version of this diagram, see the **Dynamic Geometry Exploration** Changing Triangle at www.keymath.com/DG.

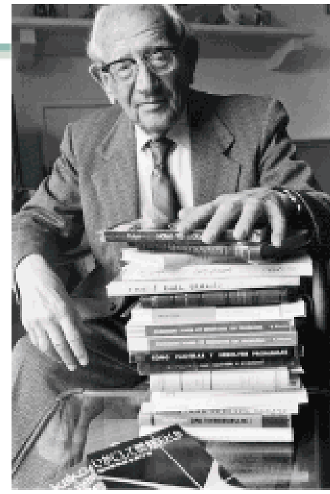
23. Find the lengths x and y. Each angle is a right angle.



keymath.com/DG

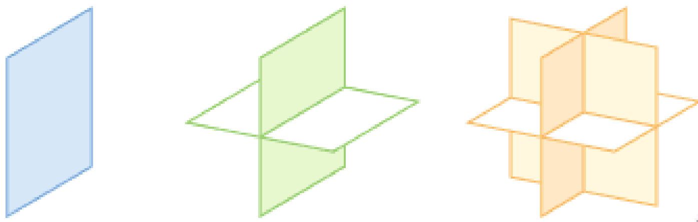
project

POLYA'S PROBLEM



George Polya (1887–1985) was a mathematician who specialized in problem-solving methods. He taught mathematics and problem solving at Stanford University for many years, and wrote the book *How to Solve It.*

He posed this problem to his students: Into how many parts will five random planes divide space?

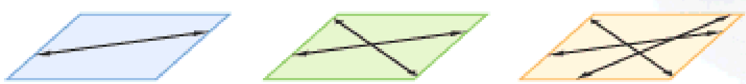


1 plane | 2 planes | 3 planes

Your project is to solve this problem. Here are some of Polya's problem-solving strategies to help you.

1. Understand the problem. Draw a figure or build a model. Can you restate the problem in your own words?

2. Break down the problem. Have you done any simpler problems that are like this one?



1 line | 2 lines | 3 lines

1 point | 2 points | 3 points

3. Check your answer. Can you find the answer in a different way to show that it is correct ? (The answer, by the way, is not 32!)

Your project should include

- All drawings and models you made or used.
- A description of the strategies you tried and how well each one worked.
- Your answer, and why you think it's correct.

For more information on Polya's Problem, go to www.keymath.com/DG .

LESSON

4.7

Flowchart Thinking

If you can only find it, there is a reason for everything.

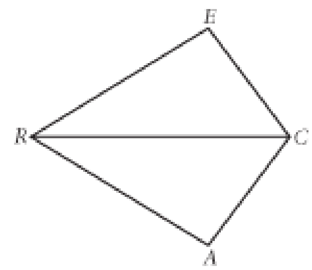
TRADITIONAL SAYING

You have been making many discoveries about triangles. As you try to explain why the new conjectures are true, you build upon definitions and conjectures you made before.

So far, you have written your explanations as deductive arguments or paragraph proofs. First, we'll look at a diagram and explain why two angles must be congruent, by writing a paragraph proof, in Example A. Then we'll look at a different tool for writing proofs, and use that tool to write the same proof, in Example B.

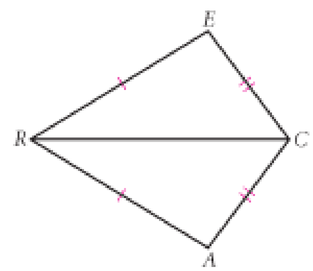
EXAMPLE A

In the figure at right, $\overline{EC} \cong \overline{AC}$ and $\overline{ER} \cong \overline{AR}$. Is $\angle A \cong \angle E$? If so, give a logical argument to explain why they are congruent.



Solution

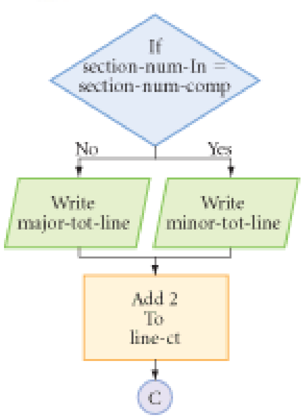
First, mark the given information on the figure. Then consider whether $\angle A$ is congruent to $\angle E$. It looks as if there are two congruent triangles, so use the reasoning strategy of applying previous conjectures to explain why.



Paragraph Proof: Show that $\angle A \cong \angle E$.

$\overline{EC} \cong \overline{AC}$ and $\overline{ER} \cong \overline{AR}$ because that information is given. $\overline{RC} \cong \overline{RC}$ because it is the same segment, and any segment is congruent to itself. So, $\triangle CRE \cong \triangle CRA$ by the SSS Congruence Conjecture. If $\triangle CRE \cong \triangle CRA$, then $\angle A \cong \angle E$ by CPCTC. ■

Career CONNECTION



Computer programmers use programming language and detailed plans to design computer software. They often use flowcharts to plan the logic in programs.

Were you able to follow the logical steps in Example A? Sometimes a logical argument or a proof is long and complex, and a paragraph might not be the clearest way to present all the steps. A **flowchart** is a visual way to organize all the steps in a complicated procedure in proper order. Arrows connect the boxes to show how facts lead to conclusions.

Flowcharts make your logic visible so that others can follow your reasoning. To present your reasoning in flowchart form, create a **flowchart proof.** Place each statement in a box. Write the logical reason for each statement beneath its box. For example, you would write "$\overline{RC} \cong \overline{RC}$, because it is the same segment," as

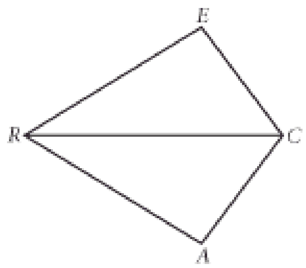
$$\boxed{\overline{RC} \cong \overline{RC}}$$

Same segment

Here is the same logical argument that you created in Example A in flowchart proof format.

EXAMPLE B

In the figure below, $\overline{EC} \cong \overline{AC}$ and $\overline{ER} \cong \overline{AR}$. Is $\angle E \cong \angle A$? If so, write a flowchart proof to explain why.

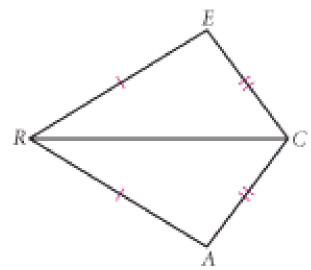


▶ Solution

First, restate the given information clearly. It helps to mark the given information on the figure. Then state what you are trying to show.

Given: $\overline{AR} \cong \overline{ER}$
$\overline{EC} \cong \overline{AC}$

Show: $\angle E \cong \angle A$



Flowchart Proof

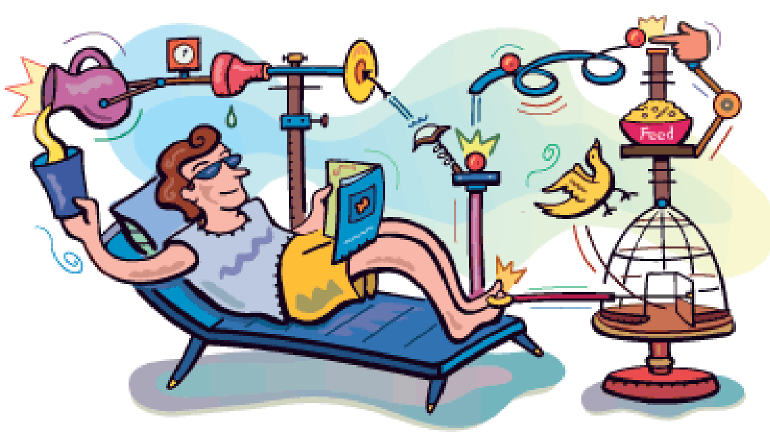
1 $\overline{AR} \cong \overline{ER}$
Given

2 $\overline{EC} \cong \overline{AC}$
Given

3 $\overline{RC} \cong \overline{RC}$
Same segment

4 $\triangle RCE \cong \triangle RCA$
SSS Congruence Conjecture

5 $\angle E \cong \angle A$
CPCTC

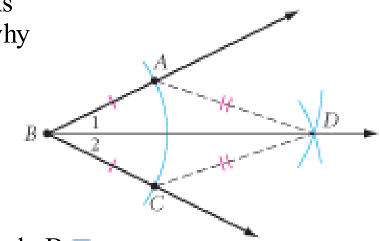


Is this contraption like a flowchart proof?

In a flowchart proof, the arrows show how the logical argument flows from the information that is given to the conclusion that you are trying to prove. Drawing an arrow is like saying "therefore." You can draw flowcharts top to bottom or left to right.

Compare the paragraph proof in Example A with the flowchart proof in Example B. What similarities and differences are there? What are the advantages of each format?

Developing Proof In Chapter 3, you learned how to construct the bisector of an angle using a compass and straightedge. Now you have the skills to explain *why* the construction works, using deductive reasoning. As a group, create a flowchart proof that explains why the construction method works.



Given: $\angle ABC$ with $\overline{BA} \cong \overline{BC}$ and $\overline{CD} \cong \overline{AD}$

Show: $\overline{BD}$ is the angle bisector of $\angle ABC$

When you are satisfied with your group's proof, discuss how it is similar and different from Example B.

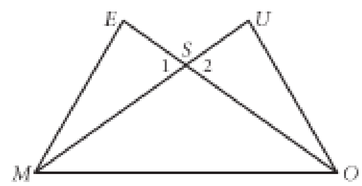
Keymath.com Practice Your Skills

Exercises

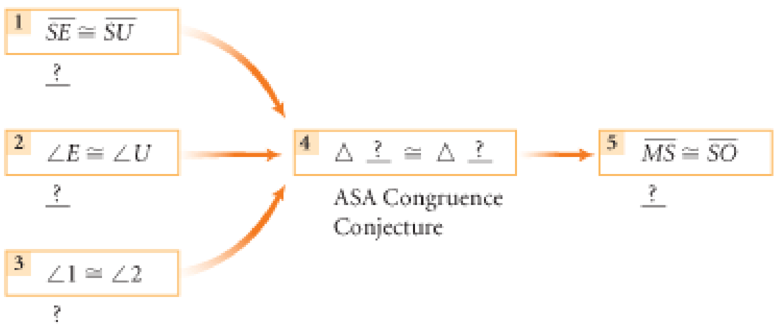
1. ***Developing Proof*** Copy the flowchart. Provide each missing reason or statement in the proof.

Given: $\overline{SE} \cong \overline{SU}$
$\angle E \cong \angle U$

Show: $\overline{MS} \cong \overline{OS}$



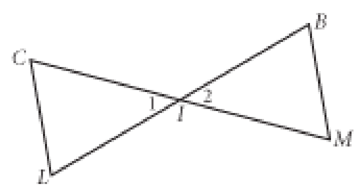
Flowchart Proof



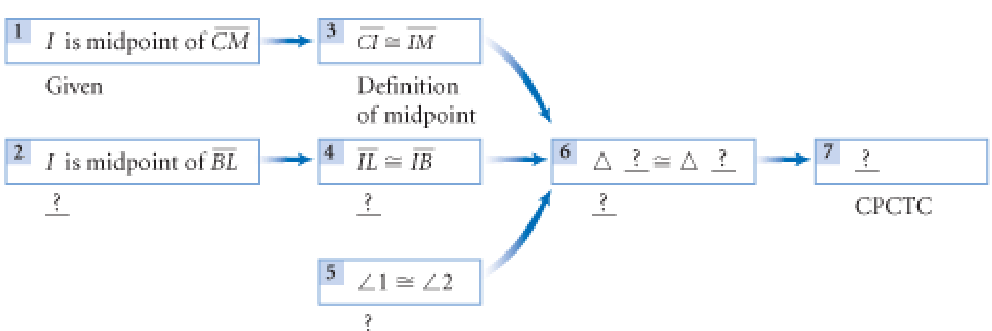
2. ***Developing Proof*** Copy the flowchart. Provide each missing reason or statement in the proof.

Given: I is the midpoint of $\overline{CM}$
I is the midpoint of $\overline{BL}$

Show: $\overline{CL} \cong \overline{MB}$



Flowchart Proof

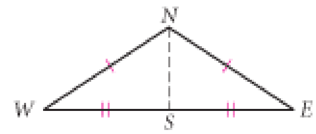


Developing Proof In Exercises 3–5, an auxiliary line segment has been added to the figure.

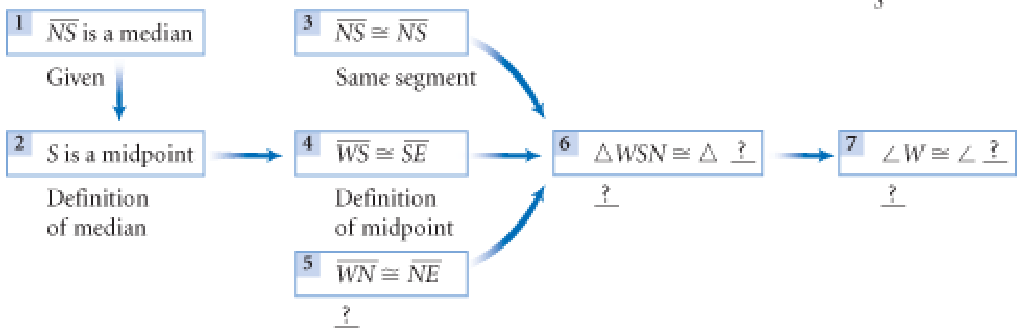
3. Complete this flowchart proof of the Isosceles Triangle Conjecture. Given that the triangle is isosceles, show that the base angles are congruent.

Given: $\triangle NEW$ is isosceles, with $\overline{WN} \cong \overline{EN}$ and median $\overline{NS}$

Show: $\angle W \cong \angle E$



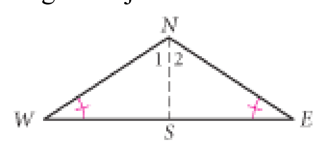
Flowchart Proof



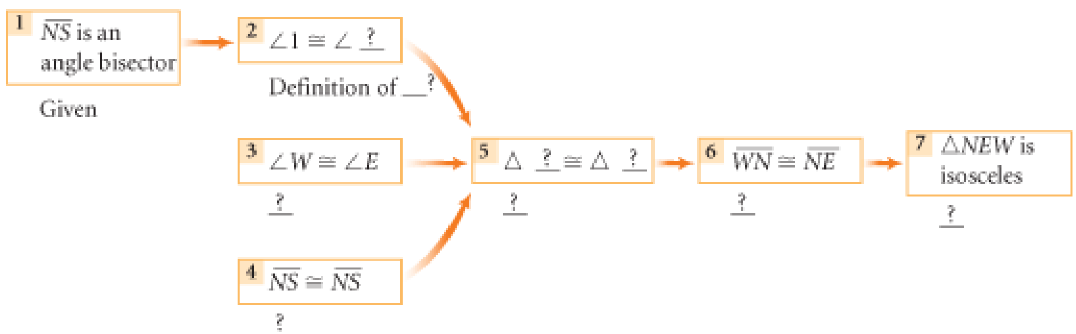
4. Complete this flowchart proof of the Converse of the Isosceles Triangle Conjecture.

Given: $\triangle NEW$ with $\angle W \cong \angle E$
$\overline{NS}$ is an angle bisector

Show: $\triangle NEW$ is an isosceles triangle



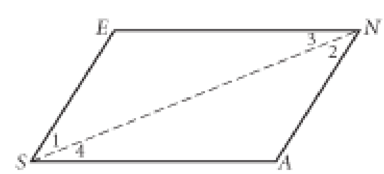
Flowchart Proof



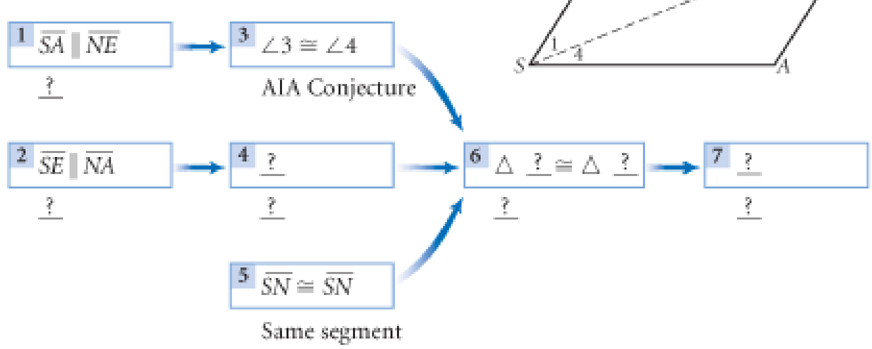
5. Complete the flowchart proof. What does this proof tell you about parallelograms?

Given: $\overline{SA} \parallel \overline{NE}$
$\overline{SE} \parallel \overline{NA}$

Show: $\overline{SA} \cong \overline{NE}$

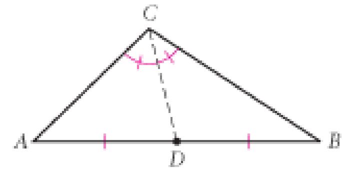


Flowchart Proof



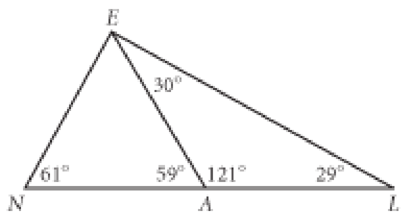
6. ***Developing Proof*** Recreate your flowchart proof from the developing proof activity on page 239 and write a paragraph proof explaining why the angle bisector construction works.

7. Suppose you saw this step in a proof: Construct angle bisector CD to the midpoint of side AB in $\triangle ABC$. What's wrong with that step? Explain.

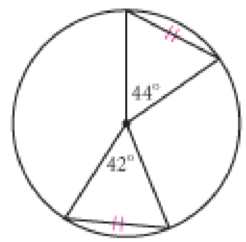


Review

8. ***Developing Proof*** Which segment is the shortest? Explain.

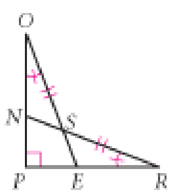


9. ***Developing Proof*** What's wrong with this picture? Explain.

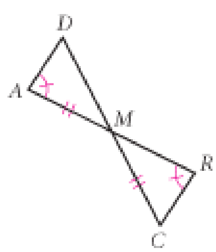


Developing Proof For Exercises 10–12, name the congruent triangles and explain why the triangles are congruent. If you cannot show that they are congruent, write "cannot be determined."

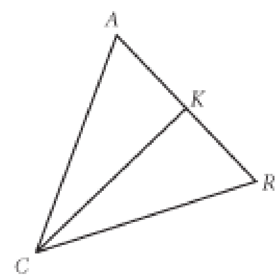
10. $\overline{PO} \cong \overline{PR}$
$\triangle POE \cong \triangle$ _?_
$\triangle SON \cong \triangle$ _?_



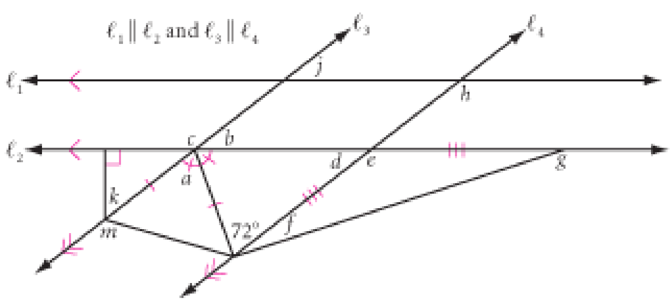
11. $\triangle$ _?_ $\cong \triangle$ _?_



12. $\overline{AC} \cong \overline{CR}$, $\overline{CK}$ is a median of $\triangle ARC$. $\triangle RCK \cong \triangle$ _?_

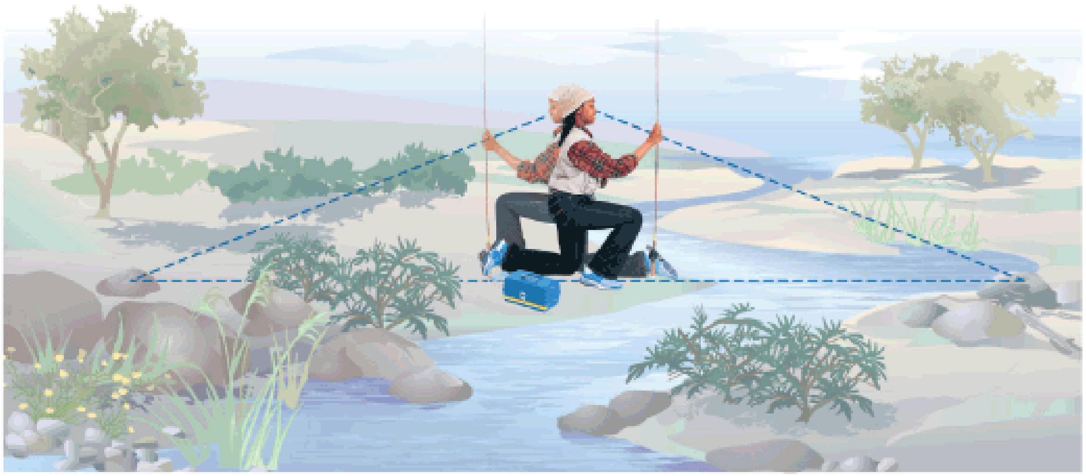


13. Copy the figure below. Calculate the measure of each lettered angle.



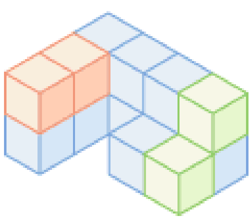
14. ***Developing Proof*** Which point of concurrency is equidistant from all three vertices? Explain why. Which point of concurrency is equidistant from all three sides? Explain why.

15. Samantha is standing at the bank of a stream, wondering how wide the stream is. Remembering her geometry conjectures, she kneels down and holds her fishing pole perpendicular to the ground in front of her. She adjusts her hand on the pole so that she can see the opposite bank of the stream along her line of sight through her hand. She then turns, keeping a firm grip on the pole, and uses the same line of sight to spot a boulder on her side of the stream. She measures the distance to the boulder and concludes that this equals the distance across the stream. What triangle congruence shortcut is Samantha using? Explain.

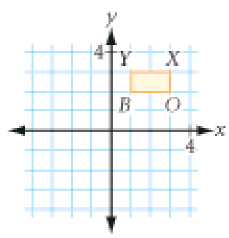


16. What is the probability of randomly selecting one of the shortest diagonals from all the diagonals in a regular decagon?

17. Sketch the solid shown with the red and green cubes removed. ⓗ



18. Sketch the new location of rectangle *BOXY* after it has been rotated 90° clockwise about the origin.



IMPROVING YOUR REASONING SKILLS

Pick a Card

Nine cards are arranged in a 3-by-3 array. Every jack borders on a king and on a queen. Every king borders on an ace. Every queen borders on a king and on an ace. (The cards border each other edge-to-edge, but not corner-to-corner.) There are at least two aces, two kings, two queens, and two jacks. Which card is in the center position of the 3-by-3 array?

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LESSON

4.8 Proving Special Triangle Conjectures

The right angle from which to approach any problem is the try angle.

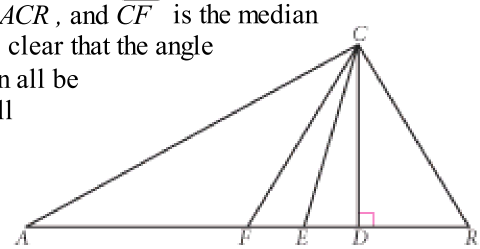
ANONYMOUS

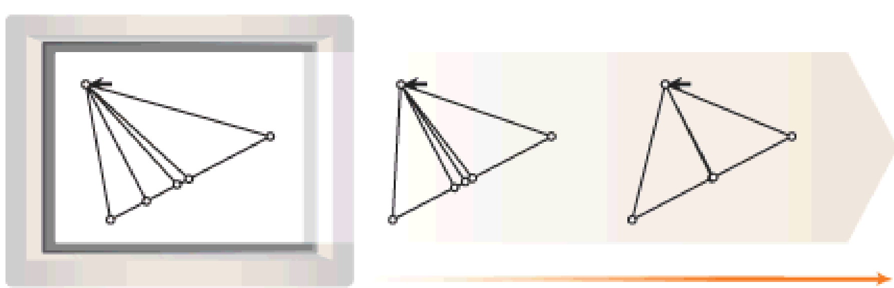
This boathouse is a remarkably symmetric structure with its isosceles triangle roof and the identical doors on each side. The rhombus-shaped attic window is centered on the line of symmetry of this face of the building. What might this building reveal about the special properties of the line of symmetry in an isosceles triangle?

In this lesson you will investigate a special segment in isosceles triangles.

In the exercises, you will prove your conjectures.

First, consider a scalene triangle. In $\triangle ARC$, $\overline{CD}$ is the altitude to the base $\overline{AR}$, $\overline{CE}$ is the angle bisector of $\angle ACR$, and $\overline{CF}$ is the median to the base $\overline{AR}$. From this example it is clear that the angle bisector, the altitude, and the median can all be different line segments. Is this true for all triangles? Can two of these ever be the same segment? Can they all be the same segment? Let's investigate.





keymath.com/DG

For an interactive version of this sketch, see the **Dynamic Geometry Exploration** The Symmetry Line in an Isosceles Triangle at www.keymath.com/DG.

Investigation

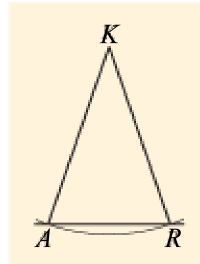
The Symmetry Line in an Isosceles Triangle

You will need

- a compass
- a straightedge

Each person in your group should draw a different isosceles triangle for this investigation.

Step 1 Construct a large isosceles triangle on a sheet of paper. Label it ARK, with K the vertex angle.



Step 2 Construct angle bisector $\overline{KD}$ with point D on $\overline{AR}$. Do $\triangle ADK$ and $\triangle RDK$ look congruent? If they are congruent, then $\overline{KD}$ is a line of symmetry.

Step 3 With your compass, compare $\overline{AD}$ and $\overline{RD}$. Is D the midpoint of $\overline{AR}$? If D is the midpoint, then what type of special segment is $\overline{KD}$?

Step 4 Compare $\angle ADK$ and $\angle RDK$. Do they have equal measures? Are they supplementary? What conclusion can you make?

Step 5 Compare your conjectures with the results of other students. Now combine the two conjectures from Steps 3 and 4 into one.

Vertex Angle Bisector Conjecture C-27

In an isosceles triangle, the bisector of the vertex angle is also ? and ?

Developing Proof In the investigation you discovered that the symmetry of an isosceles triangle yields a number of special properties. Can you explain *why* this is true for all isosceles triangles?

As a group, discuss how you would prove the Vertex Angle Bisector Conjecture. Use the reasoning strategy of drawing a labeled diagram and marking what you know. You will finish proving this conjecture in Exercises 4–6.

The properties you just discovered for isosceles triangles also apply to equilateral triangles. Equilateral triangles are also isosceles, although isosceles triangles are not necessarily equilateral.

You have probably noticed the following property of equilateral triangles: When you construct an equilateral triangle, each angle measures 60°. If each angle measures 60°, then all three angles are congruent. So, if a triangle is equilateral, then it is equiangular. This is called the Equilateral Triangle Conjecture.

If we agree that the Isosceles Triangle Conjecture is true, we can write the paragraph proof on the next page.

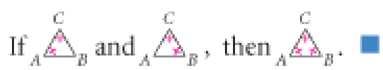
Paragraph Proof: The Equilateral Triangle Conjecture

We need to show that if $AB = AC = BC$, then $\triangle ABC$ is equiangular. By the Isosceles Triangle Conjecture,

If $AB = AC$, then $m\angle B = m\angle C$.

If $AB = BC$, then $m\angle A = m\angle C$.

If $m\angle A = m\angle C$ and $m\angle B = m\angle C$, *then* $m\angle A = m\angle B = m\angle C$. So, $\triangle ABC$ is equiangular.

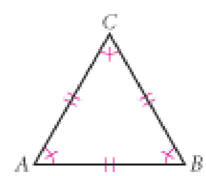


The converse of the Equilateral Triangle Conjecture is called the Equiangular Triangle Conjecture, and it states: If a triangle is equiangular, then it is equilateral. Is this true? Yes, and the proof is almost identical to the proof above, except that you use the converse of the Isosceles Triangle Conjecture. So, if the Equilateral Triangle Conjecture and the Equiangular Triangle Conjecture are both true, then we can combine them. Complete the conjecture below and add it to your conjecture list.

Equilateral/Equiangular Triangle Conjecture C-28

Every equilateral triangle is __?__, and, conversely, every equiangular triangle is __?__.

The Equilateral/Equiangular Triangle Conjecture is a **biconditional** conjecture: Both the statement and its converse are true. A triangle is equilateral *if and only if* it is equiangular. One condition cannot be true unless the other is also true.



EXERCISES

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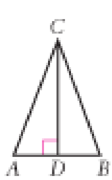
You will need

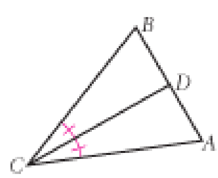
Construction tools for Exercise 11

In Exercises 1–3, $\triangle ABC$ is isosceles with $\overline{AC} \cong \overline{BC}$

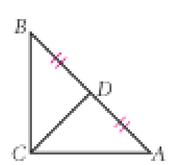
1. Perimeter $\triangle ABC = 48$ (h)
$AC = 18$
$AD =$ __?__



2. $m\angle ABC = 72°$
$m\angle ADC =$ __?__
$m\angle ACD =$ __?__

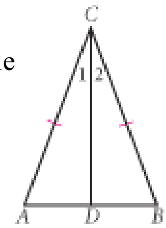


3. $m\angle CAB = 45°$
$m\angle ACD =$ __?__



Developing Proof In Exercises 4–6, copy the flowchart. Supply the missing statement and reasons in the proofs of Conjectures A, B, and C shown below. These three conjectures are all part of the Vertex Angle Bisector Conjecture.

4. Complete the flowchart proof for Conjecture A.



Conjecture A: The bisector of the vertex angle in an isosceles triangle divides the isosceles triangle into two congruent triangles.

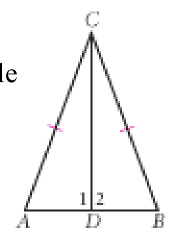
Given: $\triangle ABC$ is isosceles
$\overline{AC} \cong \overline{BC}$, and $\overline{CD}$ is the bisector of $\angle C$

Show: $\triangle ADC \cong \triangle BDC$

Flowchart Proof

1. $\overline{CD}$ is the bisector of $\angle C$ — Given
2. $\underline{?}$ — Definition of angle bisector
3. $\triangle ABC$ is isosceles with $\overline{AC} \cong \overline{BC}$ — Given
4. $\overline{CD} \cong \overline{CD}$ — Same segment
5. $\triangle ADC \cong \triangle BDC$ — $\underline{?}$

5. Complete the flowchart proof for Conjecture B.



Conjecture B: The bisector of the vertex angle in an isosceles triangle is also the altitude to the base.

Given: $\triangle ABC$ is isosceles
$\overline{AC} \cong \overline{BC}$ and $\overline{CD}$ bisects $\angle C$

Show: $\overline{CD}$ is an altitude

Flowchart Proof

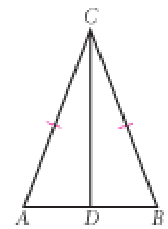
1. $\triangle ABC$ is isosceles with $\overline{AC} \cong \overline{BC}$, and $\overline{CD}$ is the bisector of $\angle C$ — Given
2. $\triangle ADC \cong \triangle BDC$ — Conjecture A
3. $\angle 1$ and $\angle 2$ form a linear pair — Definition of linear pair
4. $\angle 1 \cong \angle 2$ — $\underline{?}$
5. $\angle 1$ and $\angle 2$ are supplementary — Linear Pair Conjecture
6. $\angle 1$ and $\angle 2$ are right angles — Congruent supplementary angles are 90°
7. $\overline{CD} \perp \overline{AB}$ — $\underline{?}$
8. $\underline{?}$ — Definition of altitude

6. Create a flowchart proof for Conjecture C.

Conjecture C: The bisector of the vertex angle in an isosceles triangle is also the median to the base.

Given: $\triangle ABC$ is isosceles with $\overline{AC} \cong \overline{BC}$
$\overline{CD}$ is the bisector of $\angle C$

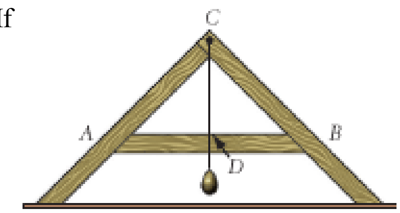
Show: $\overline{CD}$ is a median



7. ***Developing Proof*** In the figure at right, $\triangle ABC$, the plumb level is isosceles. A weight, called the plumb bob, hangs from a string attached at point C. If you place the level on a surface and the string is perpendicular to $\overline{AB}$, then the surface you are testing is level. To tell whether the string is perpendicular to $\overline{AB}$, check whether it passes through the midpoint of $\overline{AB}$. Create a flowchart proof to show that if D is the midpoint of $\overline{AB}$, then $\overline{CD}$ is perpendicular to $\overline{AB}$.

Given: $\triangle ABC$ is isosceles with $\overline{AC} \cong \overline{BC}$
D is the midpoint of $\overline{AB}$

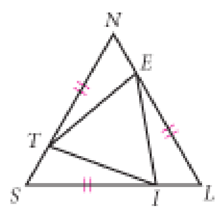
Show: $\overline{CD} \perp \overline{AB}$



History CONNECTION

Builders in ancient Egypt used a tool called a *plumb level* in building the great pyramids. With a plumb level, you can use the basic properties of isosceles triangles to determine whether a surface is level.

8. ***Developing Proof*** $\triangle SLN$ is equilateral. Is $\triangle TIE$ equilateral? Explain.



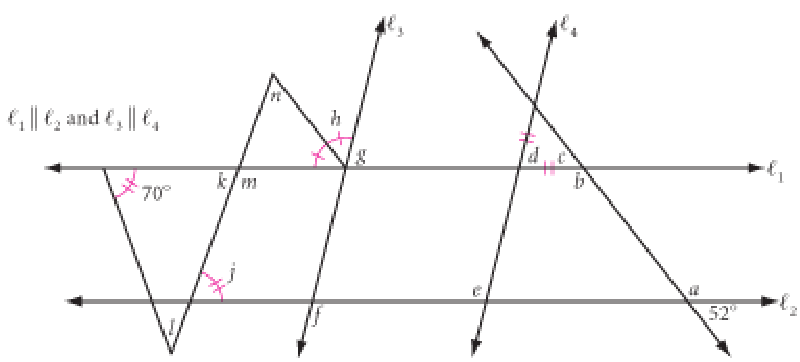
9. ***Developing Proof*** Write a paragraph proof of the Isosceles Triangle Conjecture.

10. ***Developing Proof*** Write a paragraph proof of the Equiangular Triangle Conjecture.

11. ***Construction*** Use compass and straightedge to construct a 30° angle.

Review

12. Trace the figure below. Calculate the measure of each lettered angle.



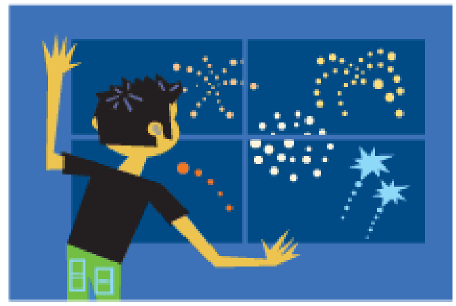
13. How many minutes after 3:00 will the hands of a clock overlap?

14. Find the equation of the line through point C that is parallel to side $\overline{AB}$ in $\triangle ABC$. The vertices are $A(1, 3)$, $B(4, -2)$, and $C(6, 6)$. Write your answer in slope-intercept form, $y = mx + b$.

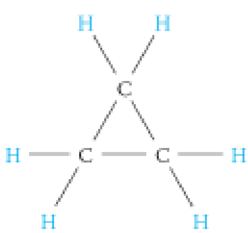
15. Sixty concurrent lines in a plane divide the plane into how many regions?

16. If two vertices of a triangle have coordinates $A(1, 3)$ and $B(7, 3)$, find the coordinates of point C so that $\triangle ABC$ is a right triangle. Can you find any other points that would create a right triangle?

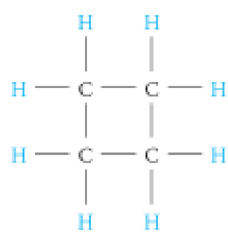
17. ***Application*** Hugo hears the sound of fireworks three seconds after he sees the flash. Duane hears the sound five seconds after he sees the flash. Hugo and Duane are 1.5 km apart. They know the flash was somewhere to the north. They also know that a flash can be seen almost instantly, but sound travels 340 m/s. Do Hugo and Duane have enough information to locate the site of the fireworks? Make a sketch and label all the distances that they know or can calculate.



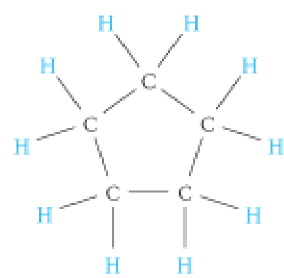
18. ***Application*** In an earlier exercise, you found the rule for the family of hydrocarbons called alkanes, or paraffins. These contain a straight chain of carbons. Alkanes can also form rings of carbon atoms. These molecules are called cycloparaffins. The first three cycloparaffins are shown below. Sketch the molecule cycloheptane. Write the general rule for cycloparaffins ($C_nH_?$).



Cyclopropane



Cyclobutane



Cyclopentane

IMPROVING YOUR ALGEBRA SKILLS

Number Tricks

Try this number trick.

Double the number of the month you were born. Subtract 16 from your answer. Multiply your result by 5, then add 100 to your answer. Subtract 20 from your result, then multiply by 10. Finally, add the day of the month you were born to your answer. The number you end up with shows the month and day you were born! For example, if you were born March 15th, your answer will be 315. If you were born December 7th, your answer will be 1207.

Number tricks almost always involve algebra. Use algebra to explain why the trick works.

Exploration

Napoleon's Theorem

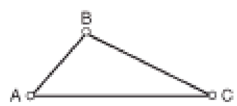
Portrait of Napoleon by the French painter Anne-Louis Girodet (1767–1824)

In this exploration you'll learn about a discovery attributed to French Emperor Napoleon Bonaparte (1769–1821). Napoleon was extremely interested in mathematics. This discovery, called Napoleon's Theorem, uses equilateral triangles constructed on the sides of any triangle.

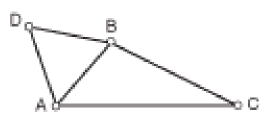
Activity

Napoleon Triangles

Step 1 Open a new Sketchpad sketch. Draw $\triangle ABC$.



Step 2 Follow the Procedure Note to create a custom tool that constructs an equilateral triangle and its centroid given the endpoints of any segment.



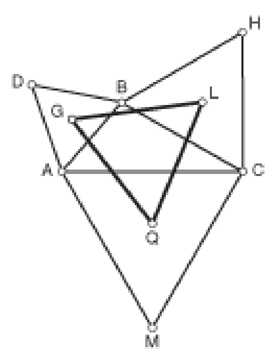
Step 3 Use your custom tool on $\overline{BC}$ and $\overline{CA}$. If an equilateral triangle falls inside your triangle, undo and try again, selecting the two endpoints in reverse order.

Step 4 Connect the centroids of the equilateral triangles. Triangle GQL is called the *outer Napoleon triangle* of $\triangle ABC$.

Drag the vertices and the sides of $\triangle ABC$ and observe what happens.

Procedure Note

1. Construct an equilateral triangle on $\overline{AB}$.
2. Construct the centroid of the equilateral triangle.
3. Hide any medians or midpoints that you constructed for the centroid.
4. Select all three vertices, all three sides, and the centroid of the equilateral triangle.
5. Turn your new construction into a custom tool by choosing **Create New Tool** from the Custom Tools menu.



Step 5 | What can you say about the outer Napoleon triangle? Write what you think Napoleon discovered in his theorem.

Here are some extensions to this theorem for you to explore.

Step 6 | Construct segments connecting each vertex of your original triangle with the vertex of the equilateral triangle on the opposite side. What do you notice about these three segments? (This discovery was made by M. C. Escher.)

Step 7 | Construct the *inner Napoleon triangle* by reflecting each centroid across its corresponding side in the original triangle. Measure the areas of the original triangle and of the outer and inner Napoleon triangles. How do these areas compare?

project

LINES AND ISOSCELES TRIANGLES

In this example, the lines $y = 3x + 3$ and $y = -3x + 3$ contain the sides of an isosceles triangle whose base is on the x-axis and whose line of symmetry is the y-axis. The window shown is $\{-4.7, 4.7, 1, -3.1, 3.1, 1\}$.

1. Find other pairs of lines that form isosceles triangles whose bases are on the x-axis and whose lines of symmetry are the y-axis.
2. Find pairs of lines that form isosceles triangles whose bases are on the y-axis and whose lines of symmetry are the x-axis.
3. A line $y = mx + b$ contains one side of an isosceles triangle whose base is on the x-axis and whose line of symmetry is the y-axis. What is the equation of the line containing the other side? Now suppose the line $y = mx + b$ contains one side of an isosceles triangle whose base is on the y-axis and whose line of symmetry is the x-axis. What is the equation of the line containing the other side?
4. Graph the lines $y = 2x - 2$, $y = \frac{1}{2}x + 1$, $y = x$, and $y = -x$. Describe the figure that the lines form. Find other sets of lines that form figures like this one.

Explore the questions above and summarize your findings. Your project should include

- Your answers to the questions above.
- A description of the patterns you found.
- Equations and graphs that are clearly labeled.

CHAPTER 4 REVIEW

In this chapter you made many conjectures about triangles. You discovered some basic properties of isosceles and equilateral triangles. You learned different ways to show that two triangles are congruent. Do you remember them all? Triangle congruence shortcuts are an important idea in geometry. You can use them to explain why your constructions work. In later chapters, you will use your triangle conjectures to investigate properties of other polygons.

You also practiced reading and writing flowchart proofs. Can you sketch a diagram illustrating each conjecture you made in this chapter? Check your conjecture list to make sure it is up to date. Make sure you have a clear diagram illustrating each conjecture.

Keymath.com Practice Your Skills

EXERCISES

You will need

Construction tools for Exercises 33, 34, and 37

1. Why are triangles so useful in structures, such as the church described in the Architecture Connection below?

2. The first conjecture of this chapter is probably the most important so far. What is it? Why do you think it is so important?

3. What special properties do isosceles triangles have?

4. What does the statement "The shortest distance between two points is the straight line between them" have to do with the Triangle Inequality Conjecture?

5. What information do you need in order to determine that two triangles are congruent? That is, what are the four congruence shortcuts?

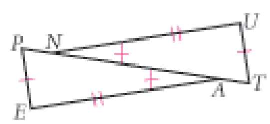
6. Explain why SSA is not a congruence shortcut.

Architecture CONNECTION

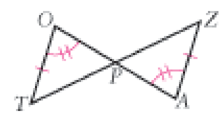
American architect Julia Morgan (1872–1957) designed many noteworthy buildings, including Hearst Castle in central California. She often used triangular trusses made of redwood and exposed beams for strength and openness, as in this church in Berkeley, California, which is now the Julia Morgan Center for the Arts. Which congruence shortcut ensures that the triangular trusses are rigid?

Developing Proof For Exercises 7–24, if possible, name the congruent triangles. State the conjecture or definition that supports the congruence statement. If you cannot show the triangles to be congruent from the information given, write "cannot be determined."

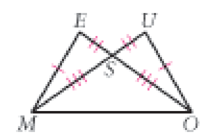
7. $\triangle PEA \cong \triangle \underline{\ ?\ }$



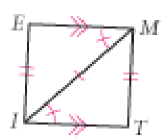
8. $\triangle TOP \cong \triangle \underline{\ ?\ }$



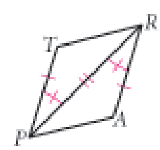
9. $\triangle MSE \cong \triangle \underline{\ ?\ }$



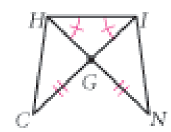
10. $\triangle TIM \cong \triangle \underline{\ ?\ }$



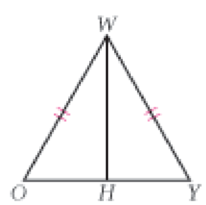
11. $\triangle TRP \cong \triangle \underline{\ ?\ }$



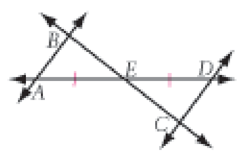
12. $\triangle CGH \cong \triangle \underline{\ ?\ }$



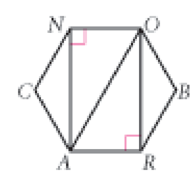
13. $\triangle \underline{\ ?\ } \cong \triangle \underline{\ ?\ }$
Is $\overline{WH}$ a median?



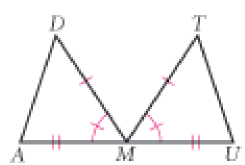
14. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$
$\triangle ABE \cong \triangle \underline{\ ?\ }$



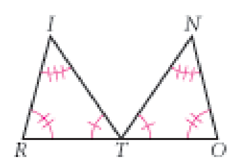
15. Polygon *CARBON* is a regular hexagon.
$\triangle ACN \cong \triangle \underline{\ ?\ }$



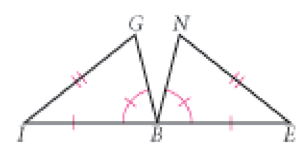
16. $\triangle \underline{\ ?\ } \cong \triangle \underline{\ ?\ }$, $\overline{AD} \cong \underline{\ ?\ }$



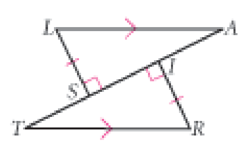
17. $\triangle \underline{\ ?\ } \cong \triangle \underline{\ ?\ }$



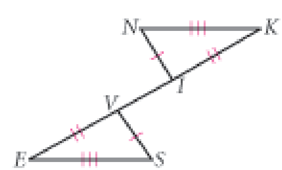
18. $\triangle \underline{\ ?\ } \cong \triangle \underline{\ ?\ }$



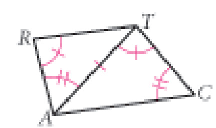
19. $\triangle \underline{\ ?\ } \cong \triangle \underline{\ ?\ }$, $\overline{TR} \cong \underline{\ ?\ }$



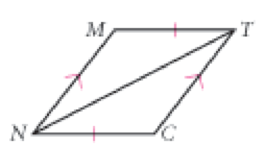
20. $\triangle \underline{\ ?\ } \cong \triangle \underline{\ ?\ }$, $\overline{EI} \cong \underline{\ ?\ }$



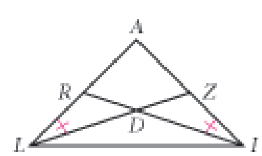
21. $\triangle CAT \cong \triangle \underline{\ ?\ }$



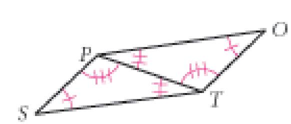
22. $\triangle \underline{\ ?\ } \cong \triangle \underline{\ ?\ }$
Is *NCTM* a parallelogram?



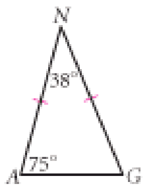
23. $\overline{IA} \cong \overline{LA}$
$\triangle \underline{\ ?\ } \cong \triangle \underline{\ ?\ }$



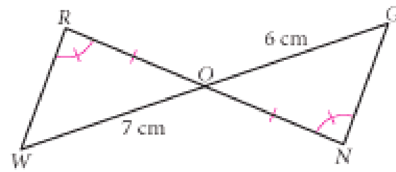
24. $\triangle \underline{\ ?\ } \cong \triangle \underline{\ ?\ }$
Is *STOP* a parallelogram?



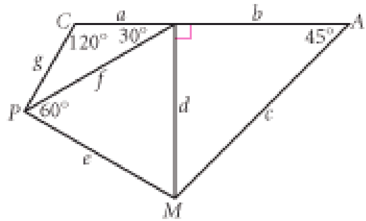
25. ***Developing Proof*** What's wrong with this picture?



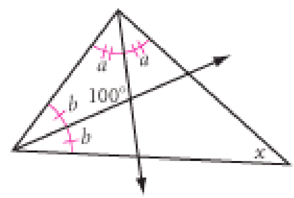
26. ***Developing Proof*** What's wrong with this pictu



27. Quadrilateral *CAMP* has been divided into three triangles. Use the angle measures provided to determine the longest and shortest segments.

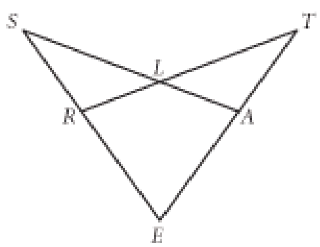


28. The measure of an angle formed by the bisectors of two angles in a triangle, as shown below, is 100°. What is angle measure x?

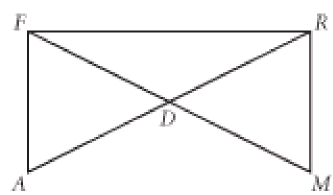


Developing Proof In Exercises 29 and 30, decide whether there is enough information to prove congruence. If there is, write a proof. If not, explain what is missing.

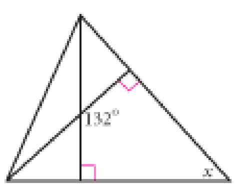
29. In the figure below, $\overline{RE} \cong \overline{AE}$, $\angle S \cong \angle T$, and $\angle ERL \cong \angle EAL$. Is $\overline{SA} \cong \overline{TR}$?



30. In the figure below, $\angle A \cong \angle M$, $\overline{AF} \perp \overline{FR}$, and $\overline{MR} \perp \overline{FR}$. Is $\triangle FRD$ isosceles?

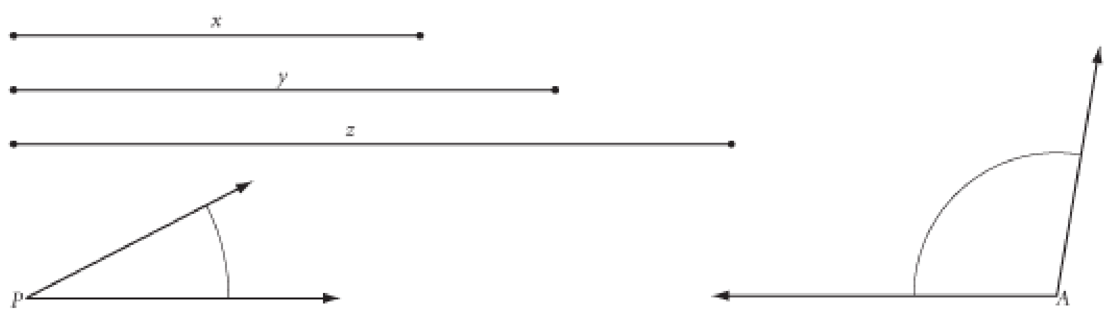


31. The measure of an angle formed by altitudes from two vertices of a triangle, as shown below, is 132°. What is angle measure x?



32. Connecting the legs of the chair at their midpoints, as shown, guarantees that the seat is parallel to the floor. Explain why.

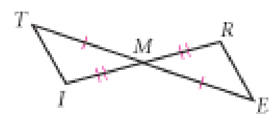
For Exercises 33 and 34, use the segments and the angles below. Use either patty paper or a compass and a straightedge. The lowercase letter above each segment represents the length of the segment.



33. ***Construction*** Construct $\triangle PAL$ given $\angle P$, $\angle A$, and $AL = y$.

34. ***Construction*** Construct two triangles $\triangle PBS$ that are not congruent to each other given $\angle P$, $PB = z$, and $SB = x$.

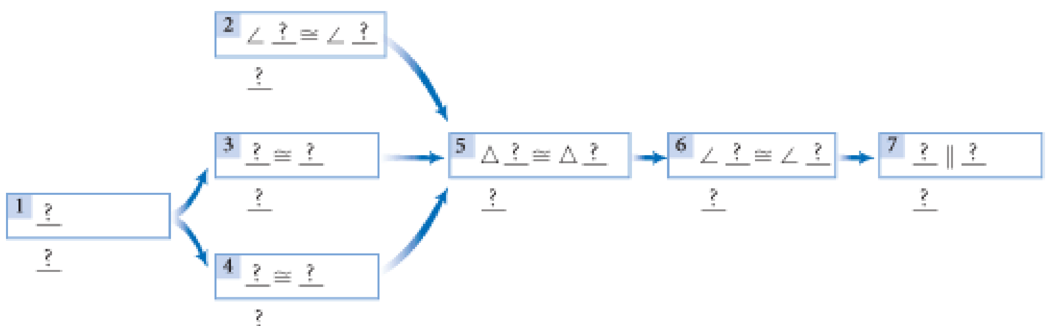
35. ***Developing Proof*** In the figure at right, is $\overline{TI} \parallel \overline{RE}$? Complete the flowchart proof or explain why they are not parallel.



Given: M is the midpoint of both $\overline{TE}$ and $\overline{IR}$

Show: $\overline{TI} \parallel \overline{RE}$

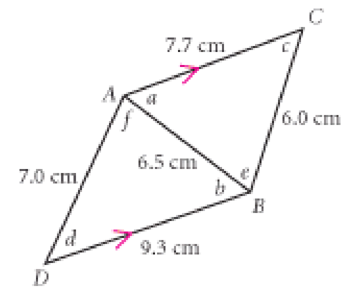
Flowchart Proof



36. At the beginning of the chapter, you learned that triangles make structures more stable. Let's revisit the shelves from Lesson 4.1. Explain how the SSS congruence shortcut guarantees that the shelves on the right will retain their shape, and why the shelves on the left wobble.

37. ***Construction*** Use patty paper or compass and straightedge to construct a 75° angle. Explain your method.

38. If $m\angle CAD > m\angle CBD$, arrange the six unknown angle measures in order from least to greatest.



TAKE ANOTHER LOOK

The section that follows, Take Another Look, gives you a chance to extend, communicate, and assess your understanding of the work you did in the investigations in this chapter. Sometimes it will lead to new, related discoveries.

1. Explore the Triangle Sum Conjecture on a sphere or a globe. Can you draw a triangle that has two or more obtuse angles? Three right angles? Write an illustrated report of your findings.

2. Investigate the Isosceles Triangle Conjecture and the Equilateral/Equiangular Triangle Conjecture on a sphere. Write an illustrated report of your findings.

3. A friend claims that if the measure of one acute angle of a triangle is half the measure of another acute angle of the triangle, then the triangle can be divided into two isosceles triangles. Try this with a computer or other tools. Describe your method and explain why it works.

4. A friend claims that if one exterior angle has twice the measure of one of the remote interior angles, then the triangle is isosceles. Use a geometry software program or other tools to investigate this claim. Describe your findings.

5. Is there a conjecture (similar to the Triangle Exterior Angle Conjecture) that you can make about exterior and remote interior angles of a convex quadrilateral? Experiment. Write about your findings.

6. Is there a conjecture you can make about inequalities among the sums of; the lengths of sides and/or diagonals of a quadrilateral? Experiment. Write about your findings.

7. ***Developing Proof*** In Chapter 3, you discovered how to construct the perpendicular bisector of a segment. Perform this construction. Now use what you've learned about congruence shortcuts to explain why this construction method works.

8. ***Developing Proof*** In Chapter 3, you discovered how to construct a perpendicular through a point on a line. Perform this construction. Use a congruence shortcut to explain why the construction works.

9. Is there a conjecture similar to the SSS Congruence Conjecture that you can make about congruence between quadrilaterals? For example, is SSSS a shortcut for quadrilateral congruence? Or, if three sides and a diagonal of one quadrilateral are congruent to the corresponding three sides and diagonal of another quadrilateral, must the two quadrilaterals be congruent (SSSD)? Investigate. Write a paragraph explaining how your conjectures follow from the triangle congruence conjectures you've learned.

Assessing What You've Learned

WRITE TEST ITEMS

It's one thing to be able to do a math problem. It's another to be able to make one up. If you were writing a test for this chapter, what would it include?

Start by having a group discussion to identify the key ideas in each lesson of the chapter. Then divide the lessons among group members, and have each group member write a problem for each lesson assigned to them. Try to create a mix of problems in your group, from simple one-step exercises that require you to recall facts to more complex, multistep problems that require more thinking. An example of a simple problem might be finding a missing angle measure in a triangle. A more complex problem could be a flowchart for a logical argument, or a word problem that requires using geometry to model a real-world situation.

Share your problems with your group members and try out one another's problems. Then discuss the problems in your group: Were they representative of the content of the chapter? Were some too hard or too easy? Writing your own problems is an excellent way to assess and review what you've learned. Maybe you can even persuade your teacher to use one of your items on a real test!

ORGANIZE YOUR NOTEBOOK Review your notebook to be sure it is complete and well organized. Write a one-page chapter summary based on your notes.

WRITE IN YOUR JOURNAL Write a paragraph or two about something you did in this class that gave you a great sense of accomplishment. What did you learn from it? What about the work makes you proud?

UPDATE YOUR PORTFOLIO Choose a piece of work from this chapter to add to your portfolio. Document the work, explaining what it is and why you chose it.

PERFORMANCE ASSESSMENT While a classmate, a friend, a family member, or your teacher observes, perform an investigation from this chapter. Explain each step, including how you arrived at the conjecture.