CHAPTER

13

Geometry as a Mathematical System

This search for new possibilities, this discovery of new jigsaw puzzle pieces, which in the first place surprises and astonishes the designer himself, is a game that through the years has always fascinated and enthralled me anew.

M. C. ESCHER

Another World (Other World), M. C. Escher, 1947

OBJECTIVES

In this chapter you will

- look at geometry as a mathematical system
- see how some conjectures are logically related to each other
- review a number of proof strategies, such as working backward and analyzing diagrams

LESSON 13.1

The Premises of Geometry

Geometry is the art of correct reasoning on incorrect figures.
GEORGE POLYA

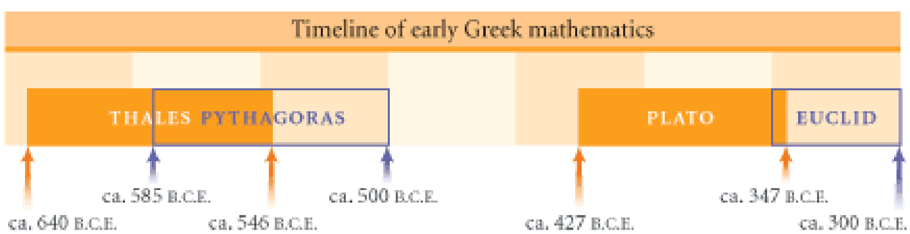
As you learned in previous chapters, for thousands of years Babylonian, Egyptian, Chinese, and other mathematicians discovered many geometry principles and developed procedures for doing practical geometry.

By 600 B.C.E., a prosperous new civilization had begun to grow in the trading towns along the coast of Asia Minor (present-day Turkey) and later in Greece, Sicily, and Italy. People had free time to discuss and debate issues of government and law. They began to insist on reasons to support statements made in debate. Mathematicians began to use logical reasoning to deduce mathematical ideas.


This map detail shows Sicily, Italy, Greece, and Asia Minor along the north coast of the Mediterranean Sea. The map was drawn by Italian painter and architect Pietro da Cortona (1596–1669).

History CONNECTION

Greek mathematician Thales of Miletus (ca. 625–547 B.C.E.) made his geometry ideas convincing by supporting his discoveries with logical reasoning. Over the next 300 years, the process of supporting mathematical conjectures with logical arguments became more and more refined. Other Greek mathematicians, including Thales' most famous student, Pythagoras, began linking chains of logical reasoning. The tradition continued with Plato and his students. Euclid, in his famous work about geometry and number theory, *Elements*, established a single chain of deductive arguments for most of the geometry known then.



You have learned that Euclid used geometric constructions to study properties of lines and shapes. Euclid also created a **deductive system**—a set of **premises**, or accepted facts, and a set of logical rules—to organize geometry properties. He started from a collection of simple and useful statements he called **postulates**. He then systematically demonstrated how each geometry discovery followed logically from his postulates and his previously proved conjectures, or **theorems**.

Up to now, you have been discovering geometry properties inductively, the way many mathematicians have over the centuries. You have studied geometric figures and have made conjectures about them. Then, to explain your conjectures, you turned to deductive reasoning. You used informal proofs to explain why a conjecture was true. However, you did not prove every conjecture. In fact, you sometimes made critical assumptions or relied on unproved conjectures in your proofs. A conclusion in a proof is true if and only if your premises are true and all your arguments are valid. Faulty assumptions can lead to the wrong conclusion. Have all your assumptions been reliable?


Inductive reasoning process

Deductive reasoning process

In this chapter you will look at geometry as Euclid did. You will start with premises: definitions, properties, and postulates. From these premises you will systematically prove your earlier conjectures. Proved conjectures will become theorems, which you can use to prove other conjectures, turning them into theorems, as well. You will build a logical framework using your most important ideas and conjectures from geometry.

Premises for Logical Arguments in Geometry

1. Definitions and undefined terms
2. Properties of arithmetic, equality, and congruence
3. Postulates of geometry
4. Previously proved geometry conjectures (theorems)

You are already familiar with the first type of premise on the list: the undefined terms—point, line, and plane. In addition, you have a list of basic definitions in your notebook.

You used the second set of premises, properties of arithmetic and equality, in your algebra course.

These Mayan stone carvings, found in Tikal, Guatemala, show the glyphs, or symbols, used in the Mayan number system. Learn more about Mayan numerals at www.keymath.com/DG .

Properties of Arithmetic

For any numbers a, b, and c:

Commutative property of addition

$a + b = b + a$

Commutative property of multiplication

$ab = ba$

Associative property of addition

$(a + b) + c = a + (b + c)$

Associative property of multiplication

$(ab)c = a(bc)$

Distributive property

$a(b + c) = ab + ac$

Properties of Equality

For any numbers a, b, c, and d:

Reflexive property

$a = a$ (Any number is equal to itself.)

Transitive property

If $a = b$ and $b = c$, then $a = c$. (This property often takes the form of the **substitution property,** which says that if $b = c$, you can substitute c for b.)

Symmetric property

If $a = b$, then $b = a$.

Addition property

If $a = b$, then $a + c = b + c$.
(Also, if $a = b$ and $c = d$, then $a + c = b + d$.)

Subtraction property

If $a = b$, then $a - c = b - c$.
(Also, if $a = b$ and $c = d$, then $a - c = b - d$.)

Multiplication property

If $a = b$, then $ac = bc$.
(Also, if $a = b$ and $c = d$, then $ac = bd$.)

Division property

If $a = b$, then $\frac{a}{c} = \frac{b}{c}$ provided $c \neq 0$.
(Also, if $a = b$ and $c = d$, then $\frac{a}{c} = \frac{b}{d}$ provided that $c \neq 0$ and $d \neq 0$.)

Square root property

If $a^2 = b$, then $a = \pm\sqrt{b}$.

Zero product property

If $ab = 0$, then $a = 0$ or $b = 0$ or both a and $b = 0$.

Whether or not you remember their names, you've used these properties to solve algebraic equations. The process of solving an equation is really an algebraic proof that your solution is valid. To arrive at a correct solution, you must support each step by a property. The addition property of equality, for example, permits you to add the same number to both sides of an equation to get an equivalent equation.

EXAMPLE Solve for x: $5x - 12 = 3(x + 2)$

▶ Solution

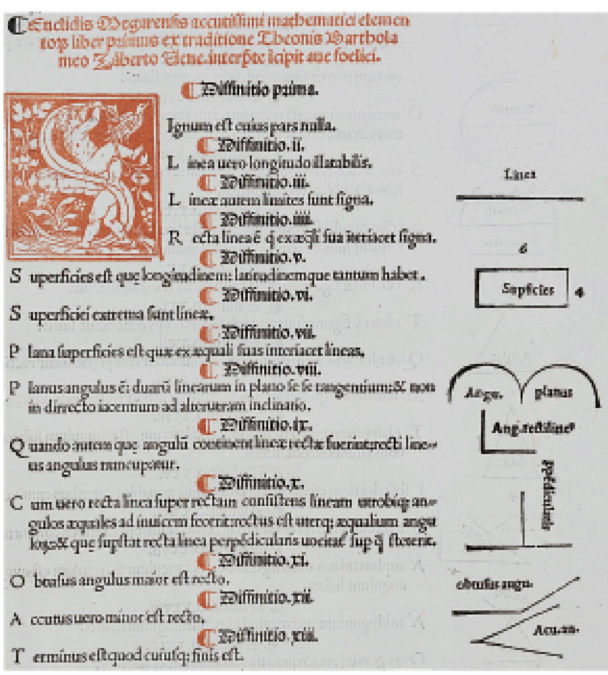
$5x - 12 = 3(x + 2)$	Given.
$5x - 12 = 3x + 6$	Distributive property.
$5x = 3x + 18$	Addition property of equality.
$2x = 18$	Subtraction property of equality.
$x = 9$	Division property of equality.

Why are the properties of arithmetic and equality important in geometry? The lengths of segments and the measures of angles involve numbers, so you will often need to use these properties in geometry proofs. And just as you use equality to express a relationship between numbers, you use congruence to express a relationship between geometric figures.

Definition of Congruence

If $AB = CD$, then $\overline{AB} \cong \overline{CD}$, and conversely, if $\overline{AB} \cong \overline{CD}$, then $AB = CD$.

If $m\angle A = m\angle B$, then $\angle A \cong \angle B$, and conversely, if $\angle A \cong \angle B$, then $m\angle A = m\angle B$.


Euclidis Megarensis acutissimi mathematici elementorum liber primus ex traditione Theonis Bartholameo Zamberto Vene. interprete incipit ane foelici.

Diffinitio prima.

Ignum est cuius pars nulla.

Diffinitio. ii.

Linea uero longitudo illatabilis.

Diffinitio. iii.

Lineae autem limites sunt signa.

Diffinitio. iiii.

Recta linea est quae ex aequo sua interiacet signa.

Diffinitio. v.

Superficies est quae longitudinem: latitudinemque tantum habet.

Diffinitio. vi.

Superficiei extrema sunt lineae.

Diffinitio. vii.

Plana superficies est quae ex aequali suas interiacet lineas.

Diffinitio. viii.

Planus angulus est: duarum linearum in plano se se tangentium: & non in directo iacentium ad alteruteram inclinatio.

Diffinitio. ix.

Quando autem quae angulum continent lineae rectae fuerint: recti lineus angulus nuncupatur.

Diffinitio. x.

Cum uero recta linea super rectam consistens lineam utrobique angulos aequales ad inuicem fecerit: rectus est uterque aequalium angulorum: & quae supstat recta linea perpendicularis uocatur super quam stererit.

Diffinitio. xi.

Obtusus angulus maior est recto.

Diffinitio. xii.

Acutus uero minor est recto.

Diffinitio. xiii.

Terminus est quod cuiusque finis est.

A page from a Latin translation of Euclid's *Elements*. Which of these definitions do you recognize?

Congruence is defined by equality, so you can extend the properties of equality to a reflexive property of congruence, a transitive property of congruence, and a symmetric property of congruence. This is left for you to do in the exercises.

The third set of premises is specific to geometry. These premises are traditionally called postulates. Postulates should be very basic. Like undefined terms, they should be useful and easy for everyone to agree on, with little debate.

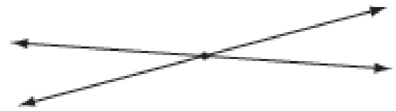
As you've performed basic geometric constructions in this class, you've observed some of these "obvious truths." Whenever you draw a figure or use an auxiliary line, you are using these postulates.

There are certain rules that everyone needs to agree on so we can drive safely! What are the "road rules" of geometry?

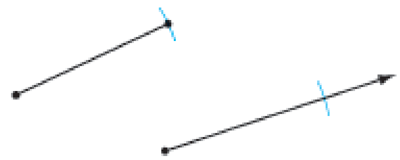
Postulates of Geometry

Line Postulate You can construct exactly one line through any two points. In other words, two points determine a line.

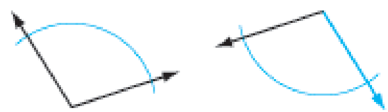
Line Intersection Postulate The intersection of two distinct lines is exactly one point.



Segment Duplication Postulate You can construct a segment congruent to another segment.

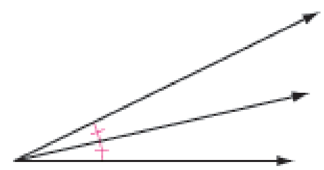


Angle Duplication Postulate You can construct an angle congruent to another angle.

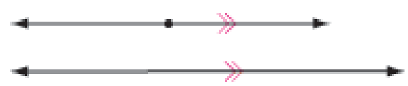


Midpoint Postulate You can construct exactly one midpoint on any line segment.

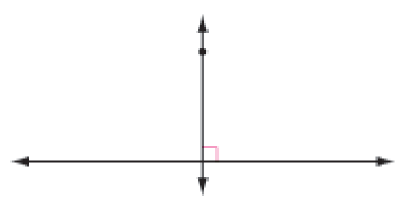
Angle Bisector Postulate You can construct exactly one angle bisector in any angle.



Parallel Postulate Through a point not on a given line, you can construct exactly one line parallel to the given line.

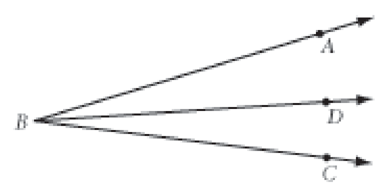


Perpendicular Postulate Through a point not on a given line, you can construct exactly one line perpendicular to the given line.

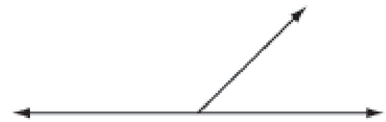


Segment Addition Postulate If point B is on $\overline{AC}$ and between points A and C, then $AB + BC = AC$.

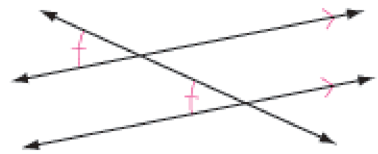
Angle Addition Postulate If point D lies in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$.



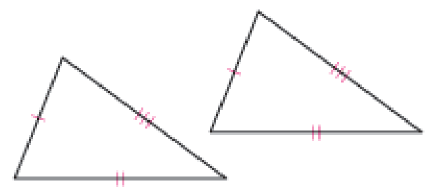
Linear Pair Postulate If two angles are a linear pair, then they are supplementary.



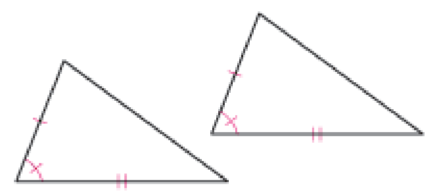
Corresponding Angles Postulate (CA Postulate) If two parallel lines are cut by a transversal, then the corresponding angles are congruent. Conversely, if two coplanar lines are cut by a transversal forming congruent corresponding angles, then the lines are parallel.



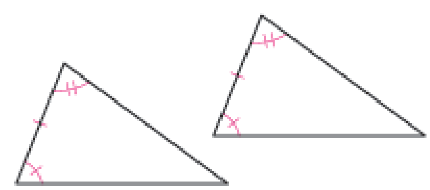
SSS Congruence Postulate If the three sides of one triangle are congruent to three sides of another triangle, then the two triangles are congruent.



SAS Congruence Postulate If two sides and the included angle in one triangle are congruent to two sides and the included angle in another triangle, then the two triangles are congruent.



ASA Congruence Postulate If two angles and the included side in one triangle are congruent to two angles and the included side in another triangle, then the two triangles are congruent.



Mathematics CONNECTION

Euclid wrote 13 books covering, among other topics, plane geometry and solid geometry. He started with definitions, postulates, and "common notions" about the properties of equality. He then wrote hundreds of propositions, which we would call conjectures, and used constructions based on the definitions and postulates to show that they were valid. The statements that we call postulates were actually Euclid's postulates, plus a few of his propositions.

To build a logical framework for the geometry you have learned, you will start with the premises of geometry. In the exercises, you will see how these premises are the foundations for some of your previous assumptions and conjectures. You will also use these postulates and properties to see how some geometry statements are logical consequences of others.

EXERCISES

Practice Your Skills

1. What is the difference between a postulate and a theorem?

2. Euclid might have stated the addition property of equality (translated from the Greek) in this way: "If equals are added to equals, the results are equal." State the subtraction, multiplication, and division properties of equality as Euclid might have stated them. (You may write them in English—extra credit for the original Greek!)

3. Write the reflexive property of congruence, the transitive property of congruence, and the symmetric property of congruence. Add these properties to your notebook. Include a diagram for each property. Illustrate one property with congruent triangles, another property with congruent segments, and another property with congruent angles. (These properties may seem ridiculously obvious. This is exactly why they are accepted as premises, which require no proof!)

4. When you state $AC = AC$, what property are you using? When you state $\overline{AC} \cong \overline{AC}$, what property are you using? (h)

5. Name the property that supports this statement: If $\angle ACE \cong \angle BDF$ and $\angle BDF \cong \angle HKM$, then $\angle ACE \cong \angle HKM$.

6. Name the property that supports this statement: If $x + 120 = 180$, then $x = 60$.

7. Name the property that supports this statement: If $2(x + 14) = 36$, then $x + 14 = 18$.

In Exercises 8 and 9, provide the missing property of equality or arithmetic as a reason for each step to solve the algebraic equation or to prove the algebraic argument.

8. Solve for x: $7x - 22 = 4(x + 2)$

Solution:	$7x - 22 = 4(x + 2)$	Given.
	$7x - 22 = 4x + 8$	? property.
	$3x - 22 = 8$	? property of equality.
	$3x = 30$	? property of equality.
	$x = 10$	? property of equality.

9. **Conjecture:** If $\frac{x}{m} - c = d$, then $x = m(c + d)$, provided that $m \neq 0$. (h)

Proof:	$\frac{x}{m} - c = d$	?
	$\frac{x}{m} = d + c$	?
	$x = m(d + c)$	?
	$x = m(c + d)$	?

In Exercises 10–17, identify each statement as true or false. Then state which definition, property of algebra, property of congruence, or postulate supports your answer.

10. If M is the midpoint of $\overline{AB}$, then $AM = BM$.

11. If M is the midpoint of $\overline{CD}$ and N is the midpoint of $\overline{CD}$, then M and N are the same point.

12. If $\overrightarrow{AB}$ bisects $\angle CAD$, then $\angle CAB \cong \angle DAB$.

13. If $\overrightarrow{AB}$ bisects $\angle CAD$ and $\overrightarrow{AF}$ bisects $\angle CAD$, then $\overrightarrow{AB}$ and $\overrightarrow{AF}$ are the same ray.

14. Lines ℓ and m can intersect at different points A and B.

15. If line ℓ passes through points A and B and line m passes through points A and B, lines ℓ and m do not have to be the same line.

16. If point P is in the interior of $\angle RAT$, then $m\angle RAP + m\angle PAT = m\angle RAT$.

17. If point M is on $\overline{AC}$ and between points A and C, then $AM + MC = AC$.

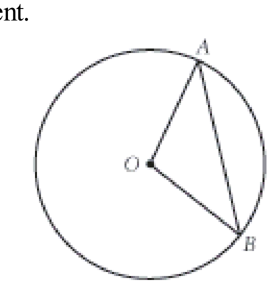
18. The Declaration of Independence states, "We hold these truths to be self-evident . . . ," then goes on to list four postulates of good government. Look up the Declaration of Independence and list the four self-evident truths that were the original premises of the United States government. You can find links to this topic at www.keymath.com/DG .

Arthur Szyk (1894–1951), a Polish American whose propaganda art helped aid the Allied war effort during World War II, created this patriotic illustrated version of the Declaration of Independence.

19. Copy and complete this flowchart proof. For each reason, state the definition, the property of algebra, or the property of congruence that supports the statement.

Given: $\overline{AO}$ and $\overline{BO}$ are radii

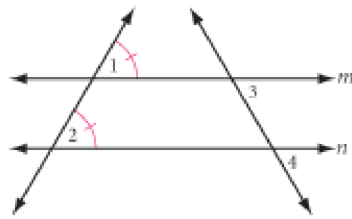
Show: $\triangle AOB$ is isosceles



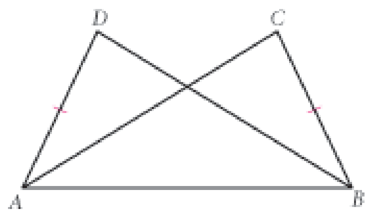
1 $\underline{?}$ (Given) → 2 $\overline{AO} \cong \overline{BO}$ (Definition of circle) → 3 $\underline{?}$ (Definition of $\underline{?}$)

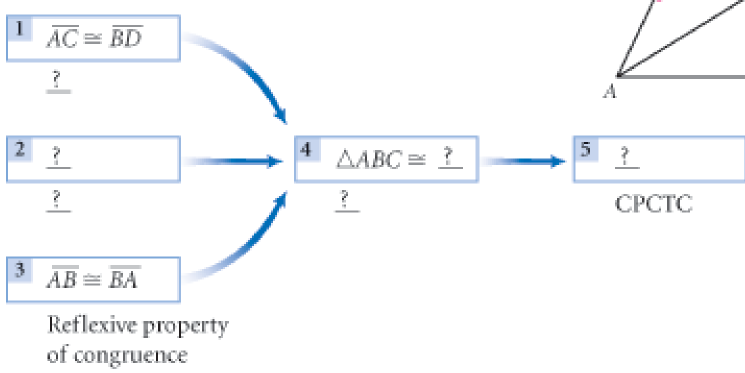
For Exercises 20–22, copy and complete each flowchart proof.

20. Given: $\angle 1 \cong \angle 2$
Show: $\angle 3 \cong \angle 4$

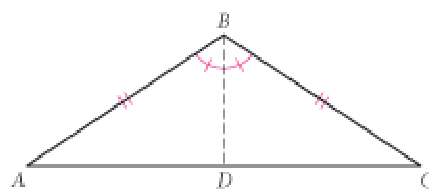


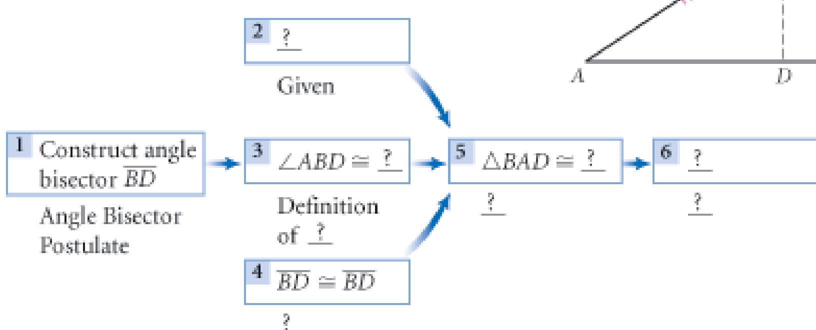
21. Given: $\overline{AC} \cong \overline{BD}$, $\overline{AD} \cong \overline{BC}$
Show: $\angle D \cong \angle C$





22. Given: Isosceles triangle ABC with $\overline{AB} \cong \overline{BC}$
Show: $\angle A \cong \angle C$





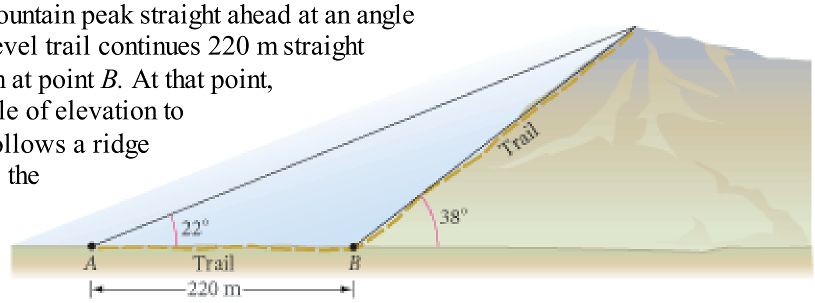
23. You have probably noticed that the sum of two odd integers is always an even integer. The rule $2n$ generates even integers and the rule $2n - 1$ generates odd integers. Let $2n - 1$ and $2m - 1$ represent any two odd integers, and prove that the sum of two odd integers is always an even integer.

24. Let $2n - 1$ and $2m - 1$ represent any two odd integers, and prove that the product of any two odd integers is always an odd integer.

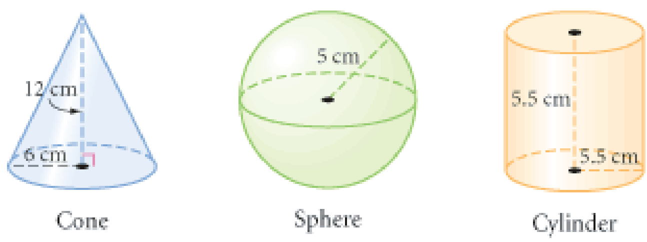
25. Show that the sum of any three consecutive integers is always divisible by 3.

Review

26. Shannon and Erin are hiking up a mountain. Of course, they are packing the clinometer they made in geometry class. At point A along a flat portion of the trail, Erin sights the mountain peak straight ahead at an angle of elevation of 22°. The level trail continues 220 m straight to the base of the mountain at point B. At that point, Shannon measures the angle of elevation to be 38°. From B the trail follows a ridge straight up the mountain to the peak. At point B, how far are they from the mountain peak?



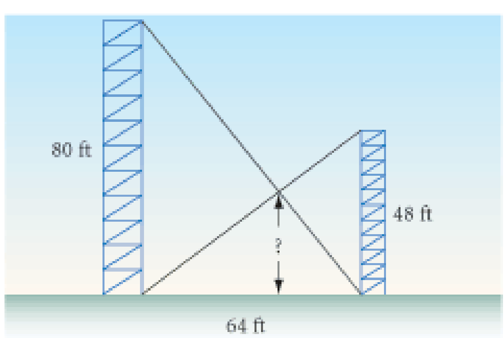
27.



Arrange the names of the solids in order, greatest to least.

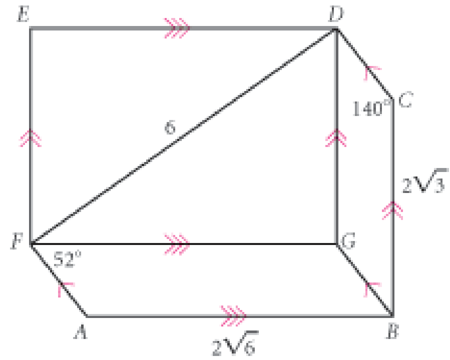
Volume:	?	?	?
Surface area:	?	?	?
Length of the longest rod that will fit inside:	?	?	?

28. Two communication towers stand 64 ft apart. One is 80 ft high and the other is 48 ft high. Each has a guy wire from its top anchored to the base of the other tower. At what height do the two guy wires cross?

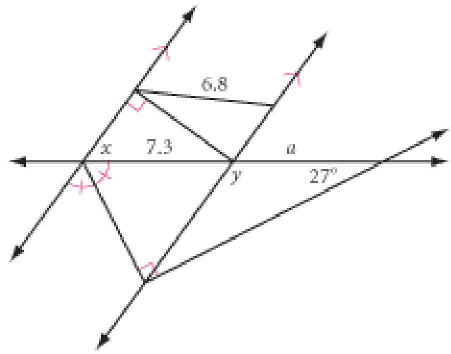


In Exercises 29 and 30, all length measurements are given in meters.

29. What's wrong with this picture?

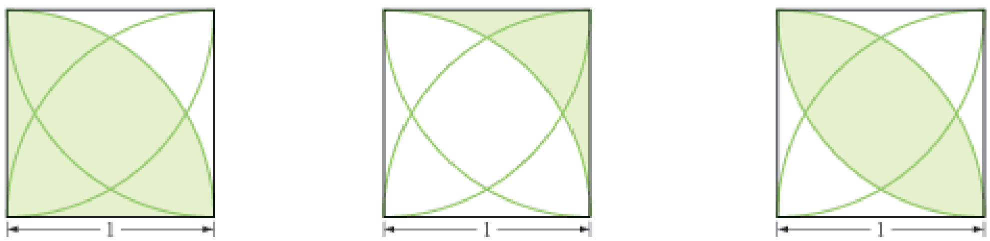


30. Find angle measures *x* and *y*, and length *a*.



31. Each arc is a quarter of a circle with its center at a vertex of the square.

Given: Each square has side length 1 unit **Find:** The shaded area



IMPROVING YOUR REASONING SKILLS

Logical Vocabulary

Here is a logical vocabulary challenge. It is sometimes possible to change one word to another of equal length by changing one letter at a time. Each change, or move, you make gives you a new word. For example, DOG can be changed to CAT in exactly three moves.

DOG ⇒ DOT ⇒ COT ⇒ CAT

Change MATH to each of the following words in exactly four moves.

1. MATH ⇒ ? ⇒ ? ⇒ ? ⇒ ROSE

2. MATH ⇒ ? ⇒ ? ⇒ ? ⇒ CORE

3. MATH ⇒ ? ⇒ ? ⇒ ? ⇒ HOST

4. MATH ⇒ ? ⇒ ? ⇒ ? ⇒ LESS

5. MATH ⇒ ? ⇒ ? ⇒ ? ⇒ LIVE

Now create one of your own. Change MATH to another word in four moves.

Student Web Links @ Keymath.com

LESSON

13.2

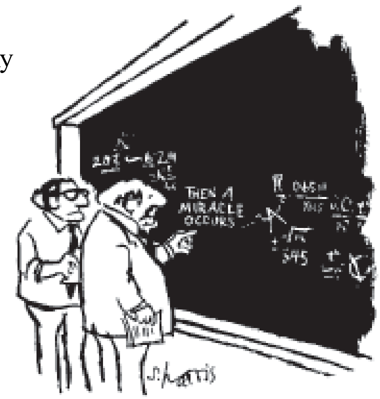
Planning a Geometry Proof

What is now proved was once only imagined.

WILLIAM BLAKE

A proof in geometry consists of a sequence of statements, starting with a given set of premises and leading to a valid conclusion. Each statement follows from one or more of the previous statements and is supported by a reason. A reason for a statement must come from the set of premises that you learned about in Lesson 13.1.

In earlier chapters you informally proved many conjectures. Now you can formally prove them, using the premises of geometry. In this lesson you will identify for yourself what is given and what you must show, in order to prove a conjecture. You will also create your own labeled diagrams.



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

©1977 by Sidney Harris, American Scientist Magazine.

As you have seen, you can state many geometry conjectures as conditional statements. For example, you can write the conjecture "Vertical angles are congruent" as a conditional statement: "If two angles are vertical angles, then they are congruent." To prove that a conditional statement is true, you assume that the first part of the conditional is true, then logically demonstrate the truth of the conditional's second part. In other words, you demonstrate that the first part implies the second part. The first part is what you assume to be true in the proof; it is the *given* information. The second part is the part you logically demonstrate in the proof; it is what you want to *show.*

Given	Show
Two angles are vertical angles	They are congruent

Next, draw and label a diagram that illustrates the given information. Then use the labels in the diagram to restate graphically what is given and what you must show.

Once you've created a diagram to illustrate your conjecture and you know where to start and where to go, make a plan using the reasoning strategies you've been developing. Use your plan to write the proof. Here's the complete process.

Writing a Proof

Task 1 From the conditional statement, identify what is given and what you must show.

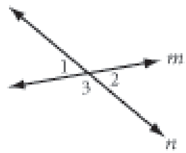
Task 2 Draw and label a diagram to illustrate the given information.

Task 3 Restate what is given and what you must show in terms of your diagram.

Task 4 Plan a proof using your reasoning strategies. Organize your reasoning mentally or on paper.

Task 5 From your plan, write a proof.

In Chapter 2, you proved the Vertical Angles Conjecture using conjectures that have now become postulates.



Lines m and n intersect to form vertical angles 1 and 2
Given

→ $\angle 1$ and $\angle 3$ are supplementary
$\angle 3$ and $\angle 2$ are supplementary
Linear Pair Postulate

→ $m\angle 1 + m\angle 3 = 180°$
$m\angle 3 + m\angle 2 = 180°$
Definition of supplementary

→ $m\angle 1 + m\angle 3 = m\angle 3 + m\angle 2$
Transitive property of equality

↓ $m\angle 1 = m\angle 2$
Subtraction property of equality

↓ $\angle 1 \cong \angle 2$
Definition of congruence

Start a theorem list separate from your conjecture list.

So the Vertical Angles Conjecture becomes the Vertical Angles (VA) Theorem. It is important when building your mathematical system that you use only the premises of geometry. These include theorems, but not unproved conjectures. You can use all the theorems on your theorem list as premises for proving other theorems. For instance, in Example A you can use the VA Theorem to prove another theorem.

You may have noticed that in the previous lesson we stated the CA Conjecture as a postulate, but not the AIA Conjecture or the AEA Conjecture. In this first example you will see how to use the five tasks of the proof process to prove the AIA Conjecture.

EXAMPLE A

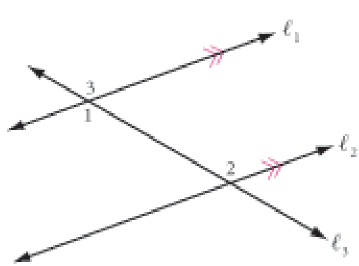
Prove the Alternate Interior Angles Conjecture: If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.

Solution

For Task 1, identify what is given and what you must show.

Given: Two parallel lines are cut by a transversal
Show: Alternate interior angles formed by the lines are congruent

For Task 2, draw and label a diagram.



For Task 3, restate what is given and what you must show in terms of the diagram.

Given: Parallel lines ℓ_1 and ℓ_2 cut by transversal ℓ_3 to form alternate interior angles $\angle 1$ and $\angle 2$

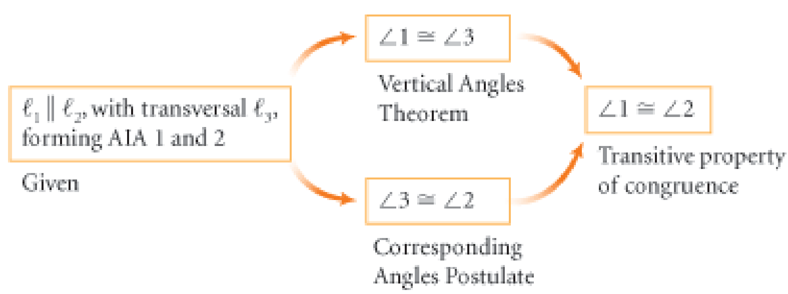
Show: $\angle 1 \cong \angle 2$

For Task 4, plan a proof. Organize your reasoning mentally or on paper.

Plan:
I need to show that $\angle 1 \cong \angle 2$.
Looking over the postulates and theorems, the ones that look useful are the CA Postulate and the VA Theorem.
From the CA Postulate, I know that $\angle 2 \cong \angle 3$ and from the VA Theorem, $\angle 1 \cong \angle 3$.
If $\angle 2 \cong \angle 3$ and $\angle 1 \cong \angle 3$, then by substitution $\angle 1 \cong \angle 2$.

For Task 5, create a proof from your plan.

Flowchart Proof



So, the AIA Conjecture becomes the AIA Theorem. Add this theorem to your theorem list.

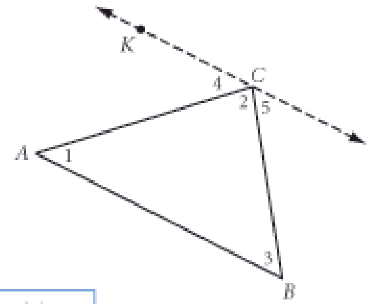
In Chapter 4, you informally proved the Triangle Sum Conjecture. The proof is short, but clever too, because it required the construction of an auxiliary line. All the steps in the proof use properties that we now designate as postulates. Example B shows the flowchart proof. For example, the Parallel Postulate guarantees that it will always be possible to construct an auxiliary line through a vertex, parallel to the opposite side.

EXAMPLE B Prove the Triangle Sum Conjecture: The sum of the measures of the angles of a triangle is 180°.

Solution

Given: $\angle 1$, $\angle 2$, and $\angle 3$ are the three angles of $\triangle ABC$

Show: $m\angle 1 + m\angle 2 + m\angle 3 = 180°$



$\angle 1$, $\angle 2$, and $\angle 3$ of $\triangle ABC$
Given

Construct $\overleftrightarrow{KC} \parallel \overline{AB}$
Parallel Postulate

$\angle 1 \cong \angle 4$
$\angle 3 \cong \angle 5$
AIA Theorem

$m\angle 1 = m\angle 4$
$m\angle 3 = m\angle 5$
Definition of congruence

$m\angle 4 + m\angle 2 = m\angle KCB$
Angle Addition Postulate

$\angle KCB$ and $\angle 5$ are supplementary
Linear Pair Postulate

$m\angle KCB + m\angle 5 = 180°$
Definition of supplementary

$m\angle 4 + m\angle 2 + m\angle 5 = 180°$
Substitution property of equality

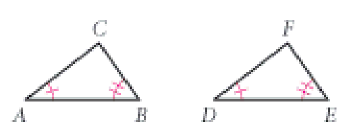
$m\angle 1 + m\angle 2 + m\angle 3 = 180°$
Substitution property of equality

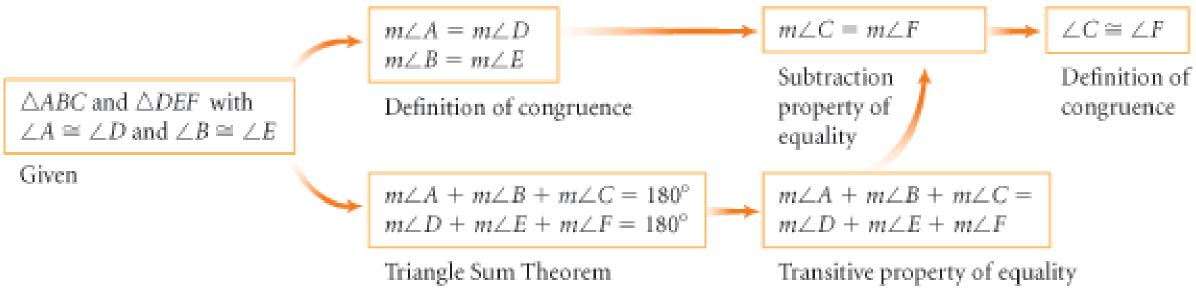
So, the Triangle Sum Conjecture becomes the Triangle Sum Theorem. Add it to your theorem list. Notice that each reason we now use in a proof is a postulate, theorem, definition, or property.

When you trace back your family tree, you include the names of your parents, the names of their parents, and so on.

To make sure a particular theorem has been properly proved, you can also check the "logical family tree" of the theorem. When you create a family tree for a theorem, you trace it back to all the postulates that the theorem relied on. You don't need to list all the definitions and properties of equality and congruence; list only the theorems and postulates used in the proof. For the theorems that were used in the proof, which postulates and theorems were used in *their* proofs, and so on. In Chapter 4, you informally proved the Third Angle Conjecture. Let's look again at the proof.

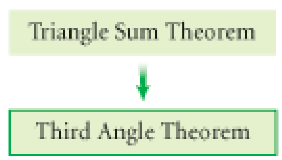
Third Angle Conjecture: If two angles of one triangle are congruent to two angles of a second triangle, then the third pair of angles are congruent.



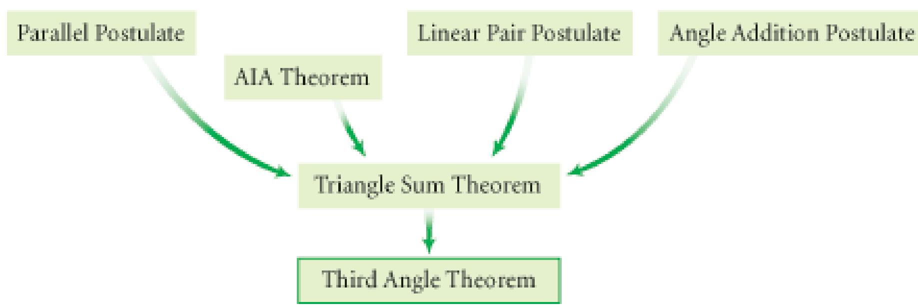


What does the logical family tree of the Third Angle Theorem look like?

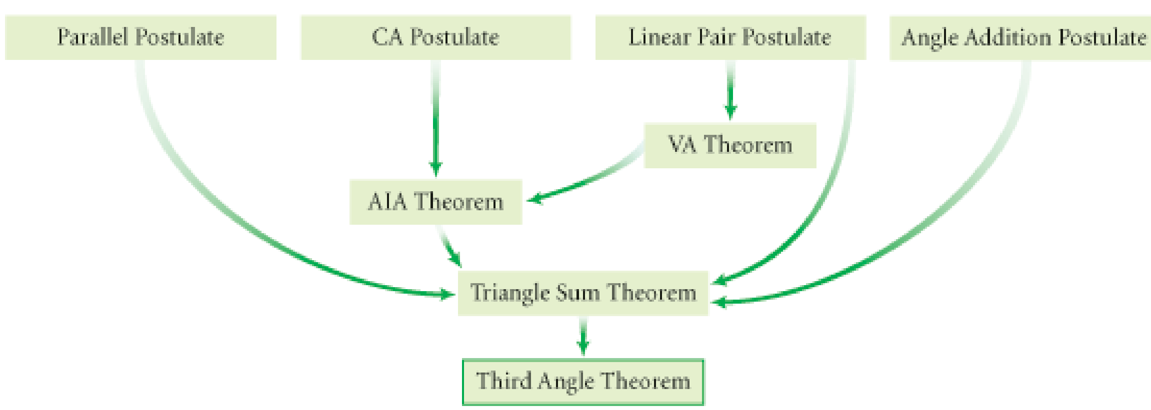
You start by putting the Third Angle Theorem in a box. Find all the postulates and theorems used in the proof. The only postulate or theorem used was the Triangle Sum Theorem. Put that box above it.



Next, locate all the theorems and postulates used to prove the Triangle Sum Theorem. Place them in boxes above the Triangle Sum Theorem. Connect the boxes with arrows showing the logical connection. Now the family tree looks like this:



To prove the AIA Theorem, we used the CA Postulate and the VA Theorem, and to prove the VA Theorem, we used the Linear Pair Postulate. The Linear Pair Postulate is already in the family tree, but move it up so it's above both the Triangle Sum Theorem and the VA Theorem. The completed family tree looks like this:



The family tree shows that, ultimately, the Third Angle Theorem relies on the Parallel Postulate, the CA Postulate, the Linear Pair Postulate, and the Angle Addition Postulate. You might notice that the family tree of a theorem looks similar to a flowchart proof. The difference is that the family tree focuses on the premises and traces them back to the postulates.

Notice how the Third Angle Theorem follows directly from the Triangle Sum Theorem without using any other theorems or postulates. A theorem that is the immediate consequence of another proven theorem is called a **corollary.** So, the Third Angle Theorem is a corollary of the Triangle Sum Theorem.

EXERCISES

1. Which postulate(s) does the VA Theorem rely on?

2. Which postulate(s) does the Triangle Sum Theorem rely on?

3. If you need a parallel line in a proof, which postulate allows you to construct it?

4. If you need a perpendicular line in a proof, which postulate allows you to construct it?

In Exercises 5–14, write a paragraph proof or a flowchart proof of the conjecture. Once you have completed their proofs, add the statements to your theorem list.

5. If two angles are both congruent and supplementary, then each is a right angle. (Congruent and Supplementary Theorem)

6. Supplements of congruent angles are congruent. (Supplements of Congruent Angles Theorem)

7. All right angles are congruent. (Right Angles Are Congruent Theorem)

8. If two lines are cut by a transversal forming congruent alternate interior angles, then the lines are parallel. (Converse of the AIA Theorem)

9. If two parallel lines are cut by a transversal, then the alternate exterior angles are congruent. (AEA Theorem)

10. If two lines are cut by a transversal forming congruent alternate exterior angles, then the lines are parallel. (Converse of the AEA Theorem)

11. If two parallel lines are cut by a transversal, then the interior angles on the same side of the transversal are supplementary. (Interior Supplements Theorem)

12. If two lines are cut by a transversal forming interior angles on the same side of the transversal that are supplementary, then the lines are parallel. (Converse of the Interior Supplements Theorem)

13. If two lines in the same plane are parallel to a third line, then they are parallel to each other. (Parallel Transitivity Theorem)

14. If two lines in the same plane are perpendicular to a third line, then they are parallel to each other. (Perpendicular to Parallel Theorem)

15. Prove that the acute angles in a right triangle are complementary. Explain why this is a corollary of the Triangle Sum Theorem.

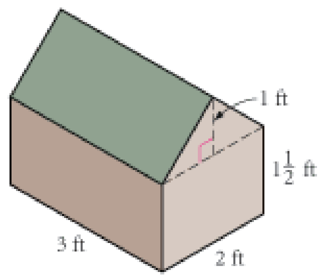
16. Draw a family tree of the Converse of the Alternate Exterior Angles Theorem.

Review

17. Suppose the top of a pyramid with volume 1107 cm^3 is sliced off and discarded, resulting in a truncated pyramid. If the cut was parallel to the base and two-thirds of the distance to the vertex, what is the volume of the truncated pyramid?

18. Abraham is building a dog house for his terrier. His plan is shown at right.

He will cut a door and a window later. After he builds the frame for the structure, can he complete it using one piece of 4-by-8-foot plywood? If the answer is yes, show how he should cut the plywood. If no, explain why not.

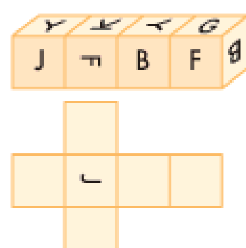


19. A triangle has vertices $A(7, -4)$, $B(3, -2)$, and $C(4, 1)$. Find the coordinates of the vertices after a dilation with center (8, 2) and scale factor 2. Complete the mapping rule for the above dilation: $(x, y) \rightarrow (\underline{?}, \underline{?})$.

IMPROVING YOUR VISUAL THINKING SKILLS

Mental Blocks

In the top figure at right, every cube is lettered exactly alike. Copy and complete the two-dimensional representation of one of the cubes to show how the letters are arranged on the six faces.



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LESSON

13.3

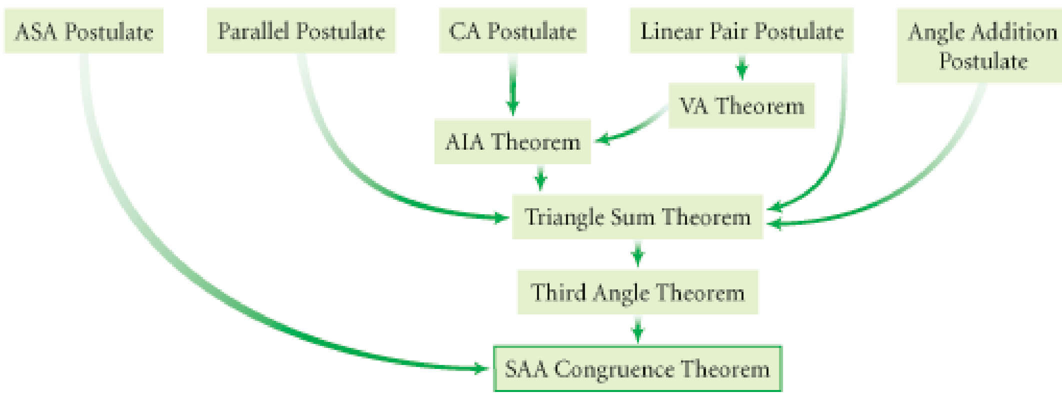
Triangle Proofs

The most violent element in our society is ignorance.

EMMA GOLDMAN

Now that the theorems from the previous lesson have been proved, make sure you have added them to your theorem list. They will be useful to you in proving future theorems.

Triangle congruence is so useful in proving other theorems that we will focus next on triangle proofs. You may have noticed that in Lesson 13.1, three of the four triangle congruence conjectures were stated as postulates (the SSS Congruence Postulate, the SAS Congruence Postulate, and the ASA Congruence Postulate). The SAA Conjecture was not stated as a postulate. In Lesson 4.5, you used the ASA Conjecture (now the ASA Postulate) to explain the SAA Conjecture. The family tree for SAA congruence looks like this:



So the SAA Conjecture becomes the SAA Theorem. Add this theorem to your theorem list. This theorem will be useful in some of the proofs in this lesson.

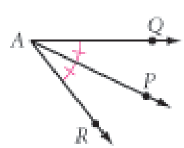
Let's use the five-task proof process and triangle congruence to prove the Angle Bisector Conjecture.

EXAMPLE Prove the Angle Bisector Conjecture: Any point on the bisector of an angle is equidistant from the sides of the angle.

Solution

Given: Any point on the bisector of an angle

Show: The point is equidistant from the sides of the angle



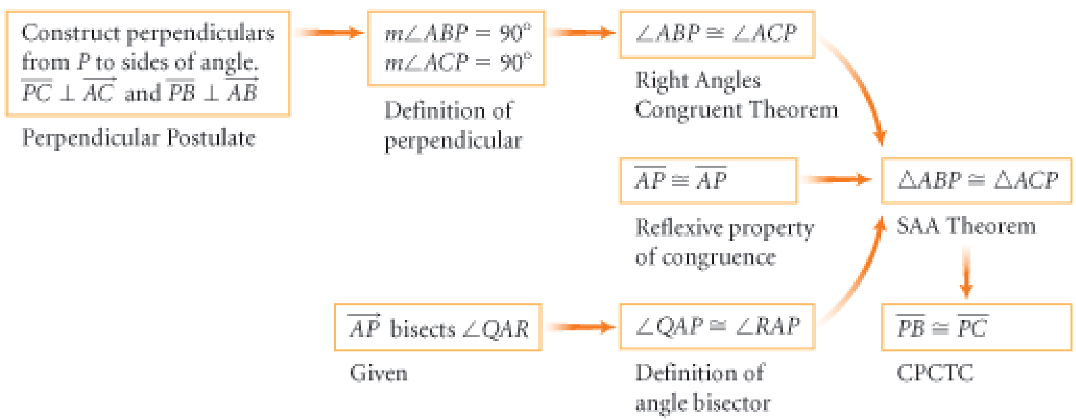
Given: $\overrightarrow{AP}$ bisecting $\angle QAR$

Show: P is equally distant from sides $\overrightarrow{AQ}$ and $\overrightarrow{AR}$

Plan: The distance from a point to a line is measured along the perpendicular from the point to the line. So I begin by constructing $\overline{PB} \perp \overrightarrow{AQ}$ and $\overline{PC} \perp \overrightarrow{AR}$ (the Perpendicular Postulate permits me to do this). I can show that $\overline{PB} \cong \overline{PC}$ if they are corresponding parts of congruent triangles. $\overline{AP} \cong \overline{AP}$ by the identity property of congruence, and $\angle QAP \cong \angle RAP$ by the definition of an angle bisector. $\angle ABP$ and $\angle ACP$ are right angles and thus they are congruent.

So $\triangle ABP \cong \triangle ACP$ by the SAA Theorem. If the triangles are congruent, then $\overline{PB} \cong \overline{PC}$ by CPCTC.

Based on this plan, I can write a flowchart proof.



Thus the Angle Bisector Conjecture becomes the Angle Bisector Theorem.

As our own proofs build on each other, flowcharts can become too large and awkward. You can also use a two-column format for writing proofs. A **two-column proof** is identical to a flowchart or paragraph proof, except that the statements are listed in the first column, each supported by a reason (a postulate, definition, property, or theorem) in the second column.

Here is the same proof from the example above, following the same plan, presented as a two-column proof. Arrows link the steps.

Statement	Reason
1. $\overrightarrow{AP}$ bisects $\angle QAR$	**1.** Given
2. $\angle QAP \cong \angle RAP$	**2.** Definition of angle bisector
3. $\overline{AP} \cong \overline{AP}$	**3.** Reflexive property of congruence
4. Construct perpendiculars from P to sides of angle so that $\overline{PC} \perp \overrightarrow{AC}$ and $\overline{PB} \perp \overrightarrow{AB}$	**4.** Perpendicular Postulate
5. $m\angle ABP = 90°$, $m\angle ACP = 90°$	**5.** Definition of perpendicular
6. $\angle ABP \cong \angle ACP$	**6.** Right Angles Congruent Theorem
7. $\triangle ABP \cong \triangle ACP$	**7.** SAA Theorem
8. $\overline{PB} \cong \overline{PC}$	**8.** CPCTC

Compare the two-column proof you just saw with the flowchart proof in Example A. What similarities do you see? What are the advantages of each format?

No matter what format you choose, your proof should be clear and easy for someone to follow.

EXERCISES

Practice Your Skills

You will need

Geometry software for Exercise 22

In Exercises 1–13, write a proof of the conjecture. Once you have completed the proofs, add the theorems to your list.

1. If a point is on the perpendicular bisector of a segment, then it is equally distant from the endpoints of the segment. (Perpendicular Bisector Theorem)

2. If a point is equally distant from the endpoints of a segment, then it is on the perpendicular bisector of the segment. (Converse of the Perpendicular Bisector Theorem) ⓗ

3. If a triangle is isosceles, then the base angles are congruent. (Isosceles Triangle Theorem)

4. If two angles of a triangle are congruent, then the triangle is isosceles. (Converse of the Isosceles Triangle Theorem)

5. If a point is equally distant from the sides of an angle, then it is on the bisector of the angle. (Converse of the Angle Bisector Theorem) ⓗ

6. The three perpendicular bisectors of the sides of a triangle are concurrent. (Perpendicular Bisector Concurrency Theorem)

7. The three angle bisectors of the sides of a triangle are concurrent. (Angle Bisector Concurrency Theorem)

8. The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles. (Triangle Exterior Angle Theorem)

9. The sum of the measures of the four angles of a quadrilateral is 360°. (Quadrilateral Sum Theorem)

10. In an isosceles triangle, the medians to the congruent sides are congruent. (Medians to the Congruent Sides Theorem)

11. In an isosceles triangle, the angle bisectors to the congruent sides are congruent. (Angle Bisectors to the Congruent Sides Theorem)

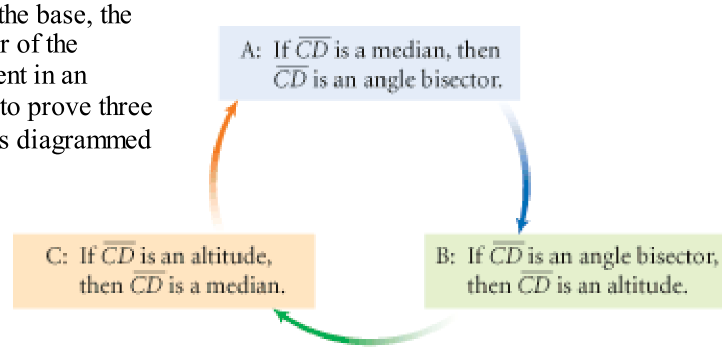
12. In an isosceles triangle, the altitudes to the congruent sides are congruent. (Altitudes to the Congruent Sides Theorem)

13. In Lesson 4.8, you were asked to complete informal proofs of these two conjectures:

The bisector of the vertex angle of an isosceles triangle is also the median to the base.

The bisector of the vertex angle of an isosceles triangle is also the altitude to the base.

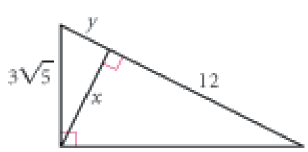
To demonstrate that the altitude to the base, the median to the base, and the bisector of the vertex angle are all the same segment in an isosceles triangle, you really need to prove three theorems. One possible sequence is diagrammed at right.



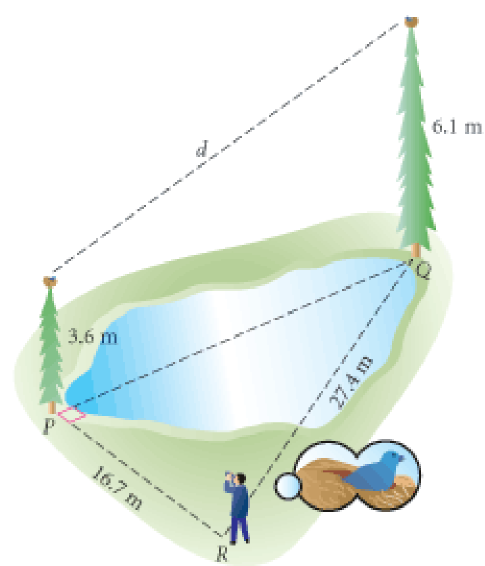
Prove the three theorems that confirm the conjecture, then add it as a theorem to your theorem list. (Isosceles Triangle Vertex Angle Theorem)

Review

14. Find x and y.



15. Two bird nests, 3.6 m and 6.1 m high, are on trees across a pond from each other, at points P and Q. The distance between the nests is too wide to measure directly (and there is a pond between the trees). A birdwatcher at point R can sight each nest along a dry path. $RP = 16.7$ m and $RQ = 27.4$ m. $\angle QPR$ is a right angle. What is the distance d between the nests?



16. Apply the glide reflection rule twice to find the first and second images of the point $A(-2, 9)$.

Glide reflection rule: A reflection across the line $x + y = 5$ and a translation $(x, y) \rightarrow (x + 4, y - 4)$.

17. Explain why $\angle 1 \cong \angle 2$.

Given:

B, G, F, E are collinear

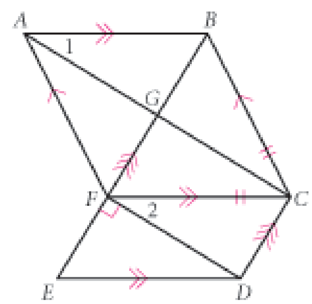
$m\angle DFE = 90°$

$BC = FC$

$\overline{AF} \parallel \overline{BC}$

$\overline{BE} \parallel \overline{CD}$

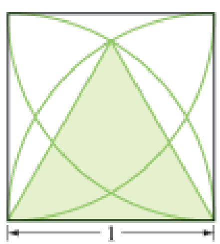
$\overline{AB} \parallel \overline{FC} \parallel \overline{ED}$



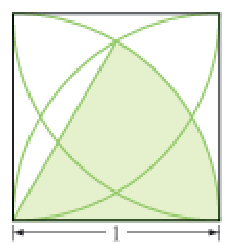
18. Each arc is a quarter of a circle with its center at a vertex of the square.

Given: The square has side length 1 unit **Find:** The shaded area

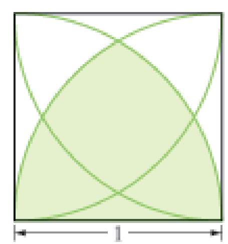
a. Shaded area = _?_



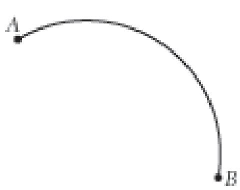
b. Shaded area = _?_



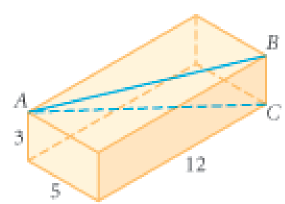
c. Shaded area = _?_



19. Given an arc of a circle on patty paper but not the whole circle or the center, fold the paper to construct a tangent at the midpoint of the arc.



20. Find $m\angle BAC$ in this right rectangular prism.



21. Choose **A** if the value of the expression is greater in Figure A.

Choose **B** if the value of the expression is greater in Figure B.

Choose **C** if the values are equal for both figures.

Choose **D** if it cannot be determined which value is greater.

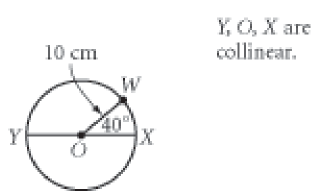


Figure A

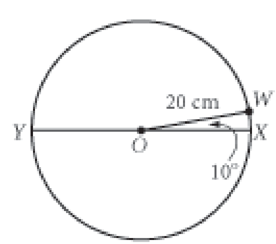


Figure B

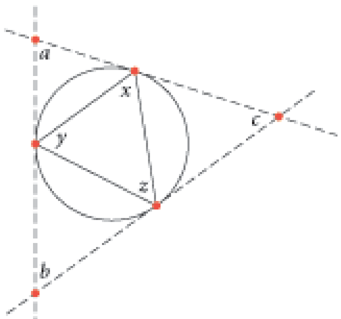
a. Perimeter of $\triangle WXY$

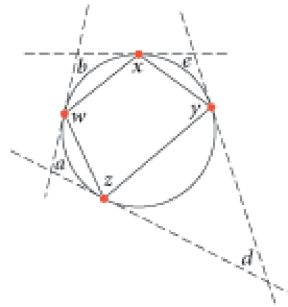
b. Area of $\triangle XOW$

22. ***Technology*** Use geometry software to construct a circle and label any three points on the circle. Construct tangents at those three points to form a circumscribed triangle and connect the points of tangency to form an inscribed triangle.

a. Drag the points and observe the angle measures of each triangle. What relationship do you notice between x, a, and c? Is the same true for y and z?

b. What is the relationship between the angle measures of a circumscribed quadrilateral and the inscribed quadrilateral formed by connecting the points of tangency?





IMPROVING YOUR REASONING SKILLS

Sudoku

Fill in every box with a digit from 1 through 9 so that every row, every column, and every 3 × 3 region contains each digit exactly once. Can you solve these puzzles logically? Find links to free online sudokus at www.keymath.com/DG .

		4				2		
1	9						5	8
		3	9		1	4		
	2			7			6	
		5	2		8	7		
	4			5			8	
		6	1		3	9		
2	7						3	1
		8				6		

4		6	8					
7				1	4	3	6	2
		3		9				
								4
		4	2		9	8		
9								
				4		7		
2	3	8	6	7				1
					3	6		5

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LESSON

13.4

Quadrilateral Proofs

All geometric reasoning is, in the last result, circular.

BERTRAND RUSSELL

In Chapter 5, you discovered and informally proved several quadrilateral properties. As reasons for the statements in some of these proofs, you used conjectures that are now postulates or that you have proved as theorems. So those steps in the proofs are valid. Occasionally, however, you may have used unproven conjectures as reasons. In this lesson you will write formal proofs of some of these quadrilateral conjectures, using only definitions, postulates, and theorems. After you have proved the theorems, you'll create a family tree tracing them back to postulates and properties.

You can prove many quadrilateral theorems by using triangle theorems. For example, you can prove some parallelogram properties by using the fact that a diagonal divides a parallelogram into two congruent triangles. In the example below, we'll prove this fact as a **lemma.** A lemma is an auxiliary theorem used specifically to prove other theorems.

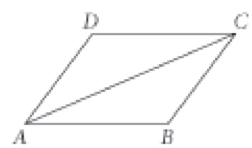
EXAMPLE

Prove: A diagonal of a parallelogram divides the parallelogram into two congruent triangles.

Solution

Given: Parallelogram $ABCD$ with diagonal $\overline{AC}$

Show: $\triangle ABC \cong \triangle CDA$



Two-Column Proof

Statement	Reason
1. $ABCD$ is a parallelogram	1. Given
2. $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$	2. Definition of parallelogram
3. $\angle CAB \cong \angle ACD$ and $\angle BCA \cong \angle DAC$	3. AIA Theorem
4. $\overline{AC} \cong \overline{AC}$	4. Reflexive property of congruence
5. $\triangle ABC \cong \triangle CDA$	5. ASA Congruence Postulate

We'll call the lemma proved in the example the Parallelogram Diagonal Lemma. You can now add it to your theorem list and use it to prove other parallelogram conjectures.

Developing Proof

Proving Parallelogram Conjectures

Work with your group to prove three of your previous conjectures about parallelograms. Remember to draw a diagram, restate what is given and what you must show in terms of your diagram, and then make a plan before you prove each conjecture.

Step 1 The Opposite Sides Conjecture states that the opposite sides of a parallelogram are congruent. Write a two-column proof of this conjecture.

Step 2 | The Opposite Angles Conjecture states that the opposite angles of a parallelogram are congruent. Write a two-column proof of this conjecture.

Step 3 | State the converse of the Opposite Sides Conjecture. Then write a two-column proof of this conjecture.

After you have successfully proved the parallelogram conjectures above, you can call them theorems and add them to your theorem list.

Step 4 | Create a family tree that shows the relationship among these theorems in Steps 1–3 and that traces each theorem back to the postulates of geometry.

EXERCISES

In Exercises 1–12, write a two-column proof or a flowchart proof of the conjecture. Once you have completed the proofs, add the theorems to your list.

1. If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram. (Converse of the Opposite Angles Theorem)

2. If one pair of opposite sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram. (Opposite Sides Parallel and Congruent Theorem)

3. Each diagonal of a rhombus bisects two opposite angles. (Rhombus Angles Theorem)

4. The consecutive angles of a parallelogram are supplementary. (Parallelogram Consecutive Angles Theorem)

5. If a quadrilateral has four congruent sides, then it is a rhombus. (Four Congruent Sides Rhombus Theorem)

6. If a quadrilateral has four congruent angles, then it is a rectangle. (Four Congruent Angles Rectangle Theorem)

7. The diagonals of a rectangle are congruent. (Rectangle Diagonals Theorem)

8. If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle. (Converse of the Rectangle Diagonals Theorem)

9. The base angles of an isosceles trapezoid are congruent. (Isosceles Trapezoid Theorem)

10. The diagonals of an isosceles trapezoid are congruent. (Isosceles Trapezoid Diagonals Theorem)

11. If a diagonal of a parallelogram bisects two opposite angles, then the parallelogram is a rhombus. (Converse of the Rhombus Angles Theorem)

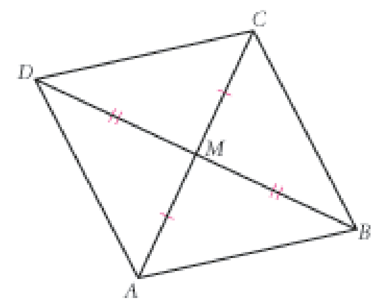
12. If two parallel lines are intersected by a second pair of parallel lines that are the same distance apart as the first pair, then the parallelogram formed is a rhombus. (Double-Edged Straightedge Theorem)

13. Create a family tree for the Parallelogram Consecutive Angles Theorem.

14. Create a family tree for the Double-Edged Straightedge Theorem.

Review

15. M is the midpoint of $\overline{AC}$ and $\overline{BD}$. For each statement, select always (A), sometimes (S), or never (N).



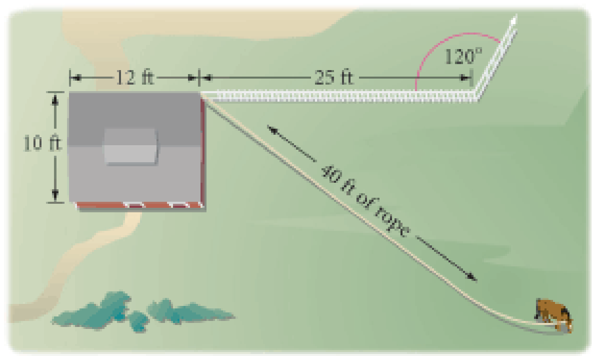
a. $\angle BAD$ and $\angle ADC$ are supplementary.

b. $\angle ADM$ and $\angle MAD$ are complementary.

c. $AD + BC < AC$

d. $AD + CD < AC$

16. Yan uses a 40 ft rope to tie his horse to the corner of the barn to which a fence is attached. How many square feet of grazing, to the nearest square foot, does the horse have?

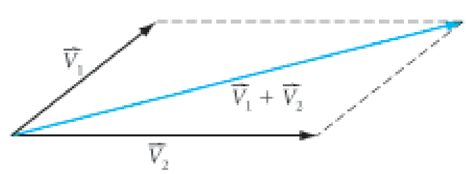


17. Complete the following chart with the symmetries and names of each type of special quadrilateral: parallelogram, rhombus, rectangle, square, kite, trapezoid, and isosceles trapezoid.

Name	Lines of symmetry	Rotational symmetry
	none	
trapezoid		
	1 diagonal	
		4-fold
	2 ⊥ bisectors of sides	
rhombus		
		none

18. Find the length and the bearing of the resultant vector $\vec{V}_1 + \vec{V}_2$

$\vec{V}_1$ has length 5 and a bearing of 40°.
$\vec{V}_2$ has length 9 and a bearing of 90°.



19. Consider the rectangular prisms in Figure A and Figure B.

Choose **A** if the value of the expression is greater in Figure A.
Choose **B** if the value of the expression is greater in Figure B.
Choose **C** if the values are equal in both rectangular prisms.
Choose **D** if it cannot be determined which value is greater.

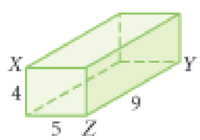


Figure A

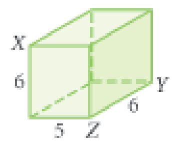


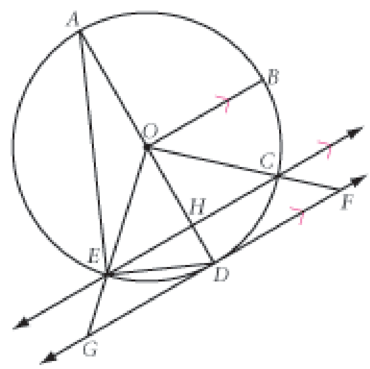
Figure B

a. Measure of $\angle XYZ$

b. Shortest path from X to Y along the surface of the prism

20. Given:

A, O, D are collinear
$\overleftrightarrow{GF}$ is tangent to circle O at point D
$m\angle EOD = 38°$
$\overline{OB} \parallel \overline{EC} \parallel \overline{GF}$



Find:

a. $m\angle AEO$

b. $m\angle DGO$

c. $m\angle BOC$

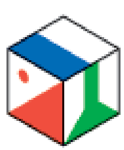
d. $m\overparen{EAB}$

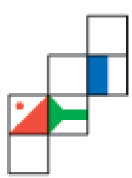
e. $m\angle HED$

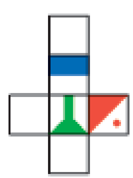
IMPROVING YOUR VISUAL THINKING SKILLS

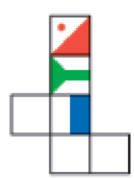
Folding Cubes II

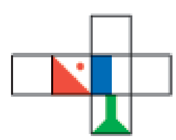
Each cube has designs on three faces. When unfolded, which figure at right could it become?

1. 

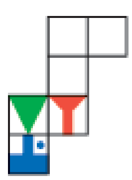
A. 

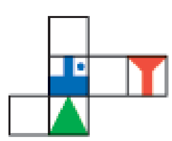
B. 

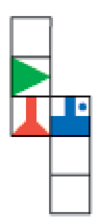
C. 

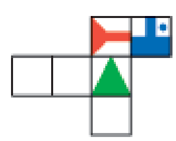
D. 

2. 

A. 

B. 

C. 

D. 

THE GEOMETER'S SKETCHPAD

Exploration

Proof as Challenge and Discovery

So far, you have proved many theorems that are useful in geometry. You can also use proof to explore and possibly discover interesting properties. You might make a conjecture, and then use proof to decide whether it is always true.

These activities have been adapted from the book *Rethinking Proof with The Geometer's Sketchpad,* 2003, by Michael deVilliers.

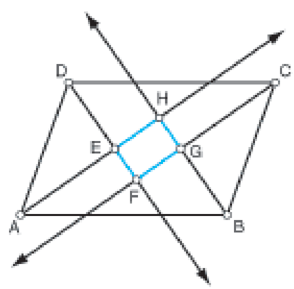
Activity

Exploring Properties of Special Constructions

Use Sketchpad to construct these figures. Drag them and notice their properties. Then prove your conjectures.

Parallelogram Angle Bisectors

Construct a parallelogram and its angle bisectors. Label your sketch as shown.



Step 1 Is *EFGH* a special quadrilateral? Make a conjecture. Drag the vertices around. Are there cases when *EFGH* does not satisfy your conjecture?

Step 2 Drag so that *ABCD* is a rectangle. What happens?

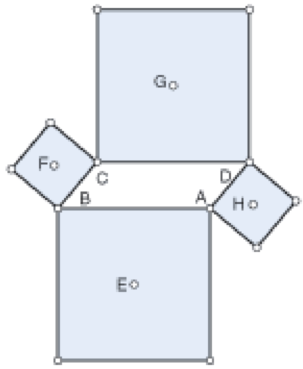
Step 3 Drag so that *ABCD* is a rhombus. What happens?

Step 4 Prove your conjectures for Steps 1–3.

Step 5 Construct the angle bisectors of another polygon. Investigate and write your observations. Make a conjecture and prove it.

Parallelogram Squares

Construct parallelogram *ABCD* and a square on each side. Construct the center of each square, and label your sketch as shown.



Step 6 Connect *E, F, G,* and *H* with line segments. (Try using a different color.) Drag the vertices of *ABCD.* What do you observe? Make a conjecture. Drag your sketch around. Are there cases when *EFGH* does not satisfy your conjecture?

Step 7 Drag so that *A, B, C,* and *D* are collinear. What happens to *EFGH*?

Step 8 Prove your conjectures for Steps 6 and 7.

Step 9 Investigate other special quadrilaterals and the shapes formed by connecting the centers of the squares on their sides. Write your observations. Make a conjecture and prove it.

IMPROVING YOUR ALGEBRA SKILLS

A Precarious Proof

You have all the money you need.

Let h = the money you have.
Let n = the money you need.

Most people think that the money they have is some amount less than the money they need. Stated mathematically, $h = n - p$ for some positive p.

If $h = n - p$, then

$$h(h - n) = (n - p)(h - n)$$
$$h^2 - hn = hn - n^2 - hp + np$$
$$h^2 - hn + hp = hn - n^2 + np$$
$$h(h - n + p) = n(h - n + p)$$

Therefore $h = n$.

So the money you have is equal to the money you need!

Is there a flaw in this proof?

LESSON

13.5

Indirect Proof

How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?

SHERLOCK HOLMES IN *THE SIGN OF THE FOUR* BY SIR ARTHUR CONAN DOYLE

In the proofs you have written so far, you have shown *directly,* through a sequence of statements and reasons, that a given conjecture is true. In this lesson you will write a different type of proof, called an indirect proof. In an **indirect proof**, you show something is true by eliminating all the other possibilities. You have probably used this type of reasoning when taking multiple-choice tests. If you are unsure of an answer, you can try to eliminate choices until you are left with only one possibility.

This mystery story gives an example of an indirect proof.

Detective Sheerluck Holmes and three other people are alone on a tropical island. One morning, Sheerluck entertains the others by playing show tunes on his ukulele. Later that day, he discovers that his precious ukulele has been smashed to bits. Who could have committed such an antimusical act? Sheerluck eliminates himself as a suspect because he knows he didn't do it. He eliminates his girlfriend as a suspect because she has been with him all day. Colonel Moran recently injured both arms and therefore could not have smashed the ukulele with such force. There is only one other person on the island who could have committed the crime. So Sheerluck concludes that the fourth person, Sir Charles Mortimer, is the guilty one.

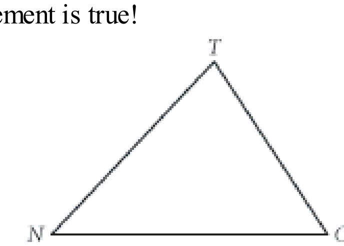
For a given mathematical statement, there are two possibilities: either the statement is true or it is not true. To prove indirectly that a statement is true, you start by assuming it is not true. You then use logical reasoning to show that this assumption leads to a contradiction. If an assumption leads to a contradiction, it must be false. Therefore, you can eliminate the possibility that the statement is not true. This leaves only one possibility—namely, that the statement is true!

EXAMPLE A

Conjecture: If $m\angle N \neq m\angle O$ in $\triangle NOT$, then $NT \neq OT$.

Given: $\triangle NOT$ with $m\angle N \neq m\angle O$

Show: $NT \neq OT$



Solution

To prove indirectly that the statement $NT \neq OT$ is true, start by assuming that it is *not* true. That is, assume $NT = OT$. Then show that this assumption leads to a contradiction.

Paragraph Proof

Assume $NT = OT$. If $NT = OT$, then $m\angle N = m\angle O$ by the Isosceles Triangle Theorem. But this contradicts the given fact that $m\angle N \neq m\angle O$. Therefore, the assumption $NT = OT$ is false and so $NT \neq OT$ is true. ■

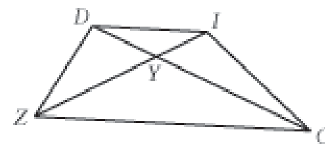
Here is another example of an indirect proof.

EXAMPLE B

Conjecture: The diagonals of a trapezoid do not bisect each other.

Given: Trapezoid $ZOID$ with parallel bases $\overline{ZO}$ and $\overline{ID}$ and diagonals $\overline{DO}$ and $\overline{IZ}$ intersecting at point Y

Show: The diagonals of trapezoid $ZOID$ do not bisect each other; that is, $DY \neq OY$ and $ZY \neq IY$



Solution

Paragraph Proof

Assume that one of the diagonals of trapezoid $ZOID$, say $\overline{ZI}$, *does* bisect the other. Then $\overline{DY} \cong \overline{OY}$. Also, by the AIA Theorem, $\angle DIY \cong \angle OZY$, and $\angle IDY \cong \angle YOZ$. Therefore, $\triangle DYI \cong \triangle OYZ$ by the SAA Theorem. By CPCTC, $\overline{ZO} \cong \overline{ID}$. It is given that $\overline{ZO} \parallel \overline{ID}$. In Lesson 13.4, you proved that if one pair of opposite sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram. So, $ZOID$ is a parallelogram. Thus, $ZOID$ has two pairs of opposite sides parallel. But because it is a trapezoid, it has exactly one pair of parallel sides. This is contradictory. Similarly, you can show that the assumption that $\overline{OD}$ bisects $\overline{ZI}$ leads to a contradiction. So the assumption that the diagonals of a trapezoid bisect each other is false and the conjecture is true. ■

In the developing proof activity you'll write an indirect proof of the Tangent Conjecture from Chapter 6.

Developing Proof

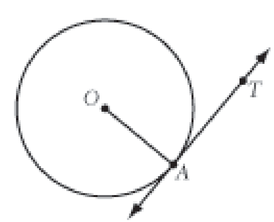
Proving the Tangent Conjecture

Copy the information and diagram below, then work with your group to complete an indirect proof of the Tangent Conjecture.

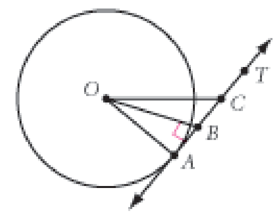
Conjecture: A tangent is perpendicular to the radius drawn to the point of tangency.

Given: Circle O with tangent $\overleftrightarrow{AT}$ and radius $\overline{AO}$

Show: $\overline{AO} \perp \overleftrightarrow{AT}$



Step 1 Assume $\overline{AO}$ is *not* perpendicular to $\overleftrightarrow{AT}$. Construct a perpendicular from point O to $\overleftrightarrow{AT}$ and label the intersection point B ($\overline{OB} \perp \overleftrightarrow{AT}$). Which postulate allows you to do this?



Step 2 Select a point C on $\overleftrightarrow{AT}$, on the other side of B from A, such that $\overline{BC} \cong \overline{AB}$. Which postulate allows you to do this?

Step 3 Next, construct $\overline{OC}$. Which postulate allows you to do this?

Step 4 $\angle ABO \cong \angle CBO$. Why?

Step 5 $\overline{OB} \cong \overline{OB}$. What property of congruence tells you this?

Step 6 Therefore, $\triangle ABO \cong \triangle CBO$. Which congruence shortcut tells you the triangles are congruent?

Step 7 If $\triangle ABO \cong \triangle CBO$, then $\overline{AO} \cong \overline{CO}$. Why?

Step 8 C must be a point on the circle (because a circle is the set of *all* points in the plane at a given distance from the center, and points A and C are both the same distance from the center). Therefore, $\overleftrightarrow{AT}$ intersects the circle in *two* points (A and C) and thus, $\overline{AT}$ is not a tangent. But this leads to a contradiction. Why?

Step 9 Discuss the steps with your group. What was the contradiction? What does it prove? Now write a complete indirect proof of the Tangent Conjecture.

Step 10 Plan and write an indirect proof of the converse of the Tangent Conjecture: A line that is perpendicular to a radius at its endpoint on the circle is tangent to the circle.

Add the Tangent Conjecture and the Converse of the Tangent Theorem to your list of theorems.

EXERCISES

For Exercises 1 and 2, the correct answer is one of the choices listed. Determine the correct answer by indirect reasoning, explaining how you eliminated each incorrect choice.

1. Which is the capital of Mali?

A. Paris **B.** Tucson **C.** London **D.** Bamako

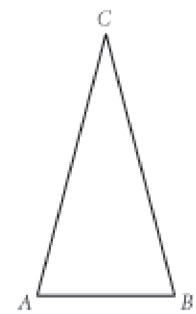
2. Which Italian scientist used a new invention called the telescope to discover the moons of Jupiter?

A. Sir Edmund Halley **B.** Julius Caesar **C.** Galileo Galilei **D.** Madonna

3. Is the proof in Example A claiming that if two angles of a triangle are not congruent, then the triangle is not isosceles? Explain.

4. Is the proof in Example B claiming that if one diagonal of a quadrilateral bisects the other, then the quadrilateral is not a trapezoid? Explain.

5. Fill in the blanks in the indirect proof below.



Conjecture: No triangle has two right angles.

Given: $\triangle ABC$

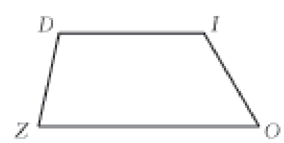
Show: No two angles are right angles

Two-Column Proof

Statement	Reason
1. Assume $\triangle ABC$ has two right angles (Assume $m\angle A = 90°$ and $m\angle B = 90°$ and $0° < m\angle C < 180°$.)	**1.** ?
2. $m\angle A + m\angle B + m\angle C = 180°$	**2.** ?
3. $90° + 90° + m\angle C = 180°$	**3.** ?
4. $m\angle C =$?	**4.** ?

But if $m\angle C = 0$, then the two sides $\overline{AC}$ and $\overline{BC}$ coincide, and thus there is no angle at C. This contradicts the given information. So the assumption is false. Therefore, no triangle has two right angles.

6. Write an indirect proof of the conjecture below.

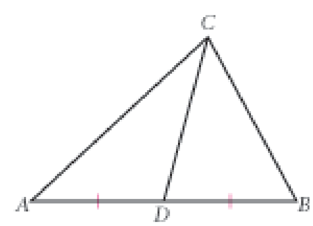


Conjecture: No trapezoid is equiangular.

Given: Trapezoid *ZOID* with bases $\overline{ZO}$ and $\overline{ID}$

Show: *ZOID* is not equiangular

7. Write an indirect proof of the conjecture below.

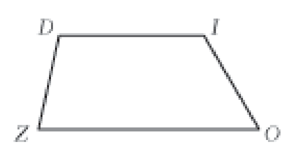


Conjecture: In a scalene triangle, the median cannot be the altitude.

Given: Scalene triangle *ABC* with median $\overline{CD}$

Show: Median $\overline{CD}$ is not the altitude to $\overline{AB}$

8. Write an indirect proof of the conjecture below.



Conjecture: The bases of a trapezoid have unequal lengths.

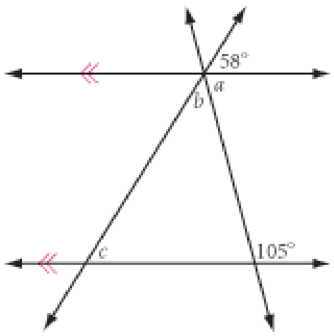
Given: Trapezoid *ZOID* with parallel bases and $\overline{ZO}$ and $\overline{ID}$

Show: $ZO \neq ID$

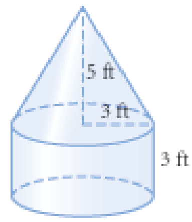
9. Write the "given" and the "show," and then plan and write the proof of the Perpendicular Bisector of a Chord Conjecture: The perpendicular bisector of a chord passes through the center of the circle. When you have finished your proof, add this to your list of theorems.

Review

10. Find *a, b,* and *c.*



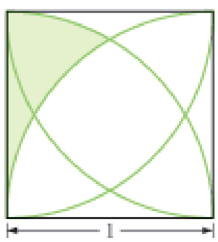
11. A clear plastic container is in the shape of a right cone atop a right cylinder, and their bases coincide. Find the volume of the container.



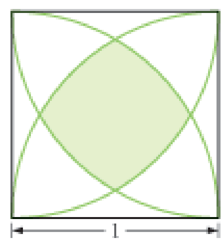
12. Each arc is a quarter of a circle with its center at a vertex of the square.

Given: The square has side length 1 unit **Find:** The shaded area

a. Shaded area = ?



b. Shaded area = ?



13. For each statement, select always (A), sometimes (S), or never (N).

a. An angle inscribed in a semicircle is a right angle.

b. An angle inscribed in a major arc is obtuse.

c. An arc measure equals the measure of its central angle.

d. The measure of an angle formed by two intersecting chords equals the measure of its intercepted arc.

e. The measure of the angle formed by two tangents to a circle equals the supplement of the central angle of the minor intercepted arc.

IMPROVING YOUR REASONING SKILLS

Symbol Juggling

If $V = \frac{1}{3}BH$, $B = \frac{1}{2}h(a + b)$, $h = 2x$, $a = 2b$, $b = x$, and $Hx = 12$, find the value of V in terms of x.

LESSON
13.6

Circle Proofs

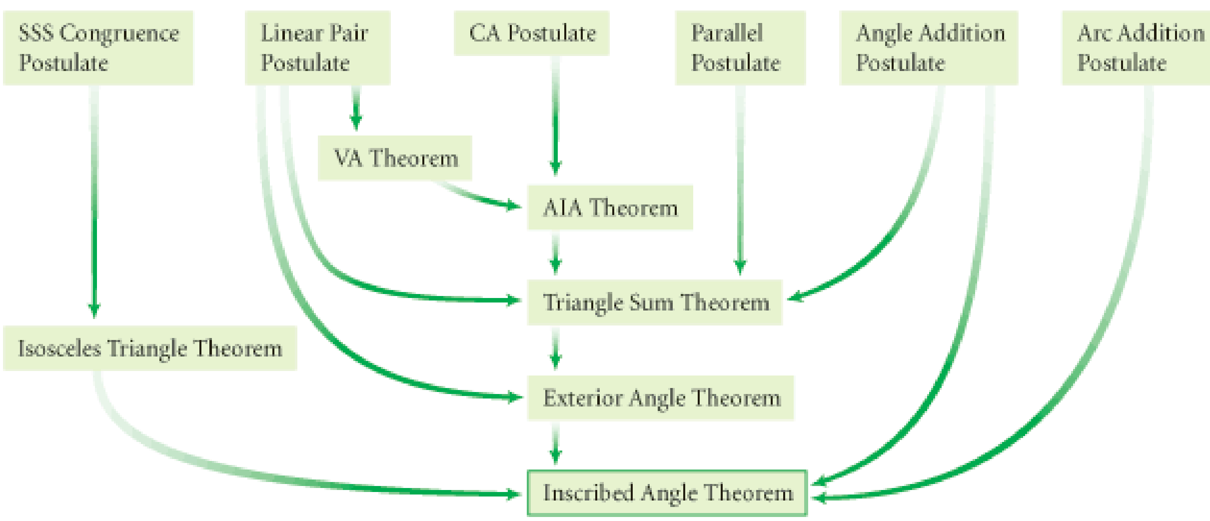
When people say, "It can't be done," or "You don't have what it takes," it makes the task all the more interesting.
LYNN HILL

In Chapter 6, you completed the proof of the three cases of the Inscribed Angle Conjecture: The measure of an inscribed angle in a circle equals half the measure of its intercepted arc. There was a lot of algebra in the proof. You may not have noticed that the Angle Addition Postulate was used, as well as a property that we called *arc addition.* Arc Addition is a postulate that you need to add to your list.

Arc Addition Postulate

If point B is on $\overset{\frown}{AC}$ and between points A and C, then $m\overset{\frown}{AB} + m\overset{\frown}{BC} = m\overset{\frown}{AC}$.

Is the proof of the Inscribed Angle Conjecture now completely supported by the premises of geometry? Can you call it a theorem? To answer these questions, trace the family tree.



So, the Inscribed Angle Conjecture is completely supported by premises of geometry; therefore you can call it a theorem and add it to your theorem list.

A double rainbow creates arcs in the sky over Stonehenge near Wiltshire, England. Built from bluestone and sandstone from 3000 to 1500 B.C.E., Stonehenge itself is laid out in the shape of a major arc.

In the exercises, you will create proofs or family trees for many of your earlier discoveries about circles.

EXERCISES

You will need

Construction tools for Exercise 17

Geometry software for Exercise 15

In Exercises 1–7, set up and write a proof of each conjecture. Once you have completed the proofs, add the theorems to your list.

1. Inscribed angles that intercept the same or congruent arcs are congruent. (Inscribed Angles Intercepting Arcs Theorem)

2. The opposite angles of an inscribed quadrilateral are supplementary. (Cyclic Quadrilateral Theorem)

3. Parallel lines intercept congruent arcs on a circle. (Parallel Secants Congruent Arcs Theorem)

4. If a parallelogram is inscribed within a circle, then the parallelogram is a rectangle. (Parallelogram Inscribed in a Circle Theorem)

5. Tangent segments from a point to a circle are congruent. (Tangent Segments Theorem)

6. The measure of an angle formed by two intersecting chords is half the sum of the measures of the two intercepted arcs. (Intersecting Chords Theorem)

7. Write and prove a theorem about the arcs intercepted by secants intersecting outside a circle, and the angle formed by the secants. (Intersecting Secants Theorem)

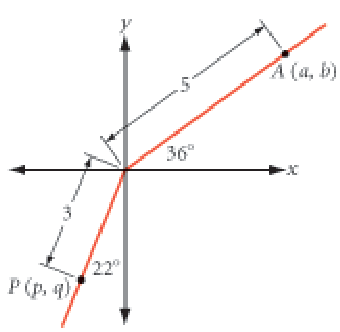
8. Prove the Angles Inscribed in a Semicircle Conjecture: An angle inscribed in a semicircle is a right angle. Explain why this is a corollary of the Inscribed Angle Theorem.

9. Create a family tree for the Tangent Segments Theorem.

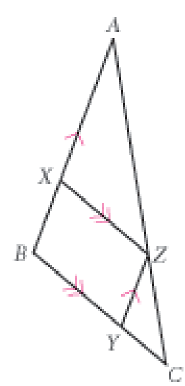
10. Create a family tree for the Parallelogram Inscribed in a Circle Theorem.

Review

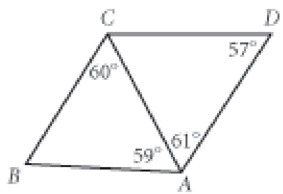
11. Find the coordinates of A and P to the nearest tenth.



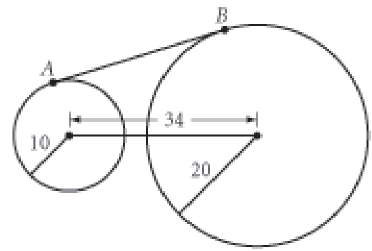
12. **Given:** $AX = 6$, $XB = 2$, $BC = 4$, $ZC = 3$
Find: $BY = \underline{?}$, $YC = \underline{?}$, $AZ = \underline{?}$



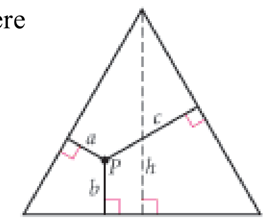
13. List the five segments in order from shortest to longest.



14. $\overline{AB}$ is a common external tangent. Find the length of $\overline{AB}$ (to a tenth of a unit).



15. ***Technology*** P is any point inside an equilateral triangle. Is there a relationship between the height h and the sum $a + b + c$? Use geometry software to explore the relationship and make a conjecture. Then write a proof of your conjecture.

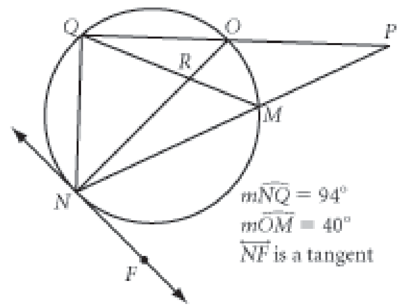


16. Find each measure or conclude that it "cannot be determined."

a. $m\angle P$ **b.** $m\angle QON$

c. $m\angle QRN$ **d.** $m\angle QMP$

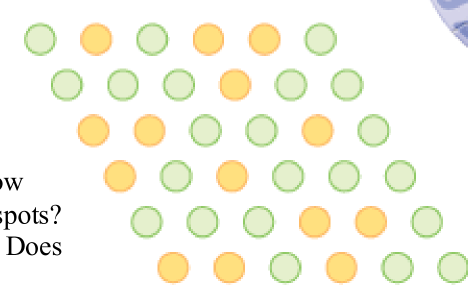
e. $m\angle ONF$ **f.** $m\overset{\frown}{MN}$



17. ***Construction*** Use a compass and straightedge to construct the two tangents to a circle from a point outside the circle.

IMPROVING YOUR REASONING SKILLS

Seeing Spots



The arrangement of green and yellow spots at right may appear to be random, but there is a pattern. Each row is generated by the row immediately above it. Find the pattern and add several rows to the arrangement. Do you think a row could ever consist of all yellow spots? All green spots? Could there ever be a row with one green spot? Does a row ever repeat itself?

LESSON

13.7

Similarity Proofs

Mistakes are part of the dues one pays for a full life.

SOPHIA LOREN

To prove conjectures involving similarity, we need to extend the properties of equality and congruence to similarity.

Properties of Similarity

Reflexive property of similarity

Any figure is similar to itself.

Symmetric property of similarity

If Figure A is similar to Figure B, then Figure B is similar to Figure A.

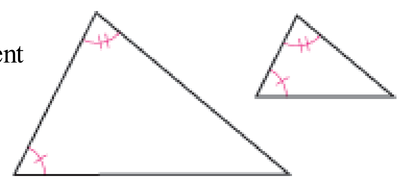
Transitive property of similarity

If Figure A is similar to Figure B and Figure B is similar to Figure C, then Figure A is similar to Figure C.

The AA Similarity Conjecture is actually a similarity postulate.

AA Similarity Postulate

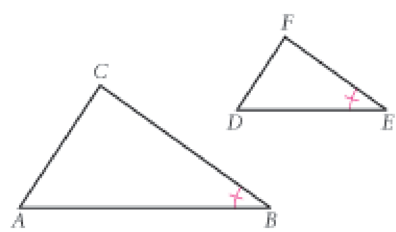
If two angles of one triangle are congruent to two angles of another triangle, then the two triangles are similar.



In Chapter 11, you also discovered the SAS and SSS shortcuts for showing that two triangles are similar. In the example that follows, you will see how to use the AA Similarity Postulate to prove the SAS Similarity Conjecture, making it the SAS Similarity Theorem.

EXAMPLE

Prove the SAS Similarity Conjecture: If two sides of one triangle are proportional to two sides of another triangle and the included angles are congruent, then the two triangles are similar.

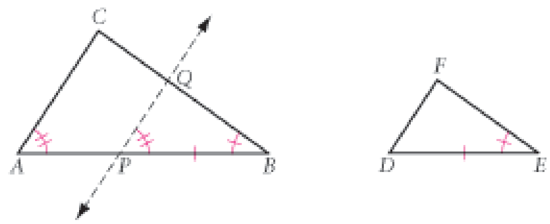


▶ Solution

Given: $\triangle ABC$ and $\triangle DEF$ such that $\frac{AB}{DE} = \frac{BC}{EF}$ and $\angle B \cong \angle E$

Show: $\triangle ABC \sim \angle DEF$

Plan: The only shortcut for showing that two triangles are similar is the AA Similarity Postulate, so you need to find another pair of congruent angles. One way of getting two congruent angles is to find two congruent triangles. You can draw a triangle within $\triangle ABC$ that is congruent to $\triangle DEF$. The Segment Duplication Postulate allows you to locate a point P on $\overline{AB}$ such that $PB = DE$. The Parallel Postulate allows you to construct a line $\overleftrightarrow{PQ}$ parallel to $\overline{AC}$. Then $\angle A \cong \angle QPB$ by the CA Postulate.



Now, if you can show that $\triangle PBQ \cong \triangle DEF$, then you will have two congruent pairs of angles to prove $\triangle ABC \sim \triangle DEF$. So, how do you show that $\triangle PBQ \cong \triangle DEF$? If you can get $\triangle ABC \sim \triangle PBQ$, then $\frac{AB}{PB} = \frac{BC}{BQ}$. It is given that $\frac{AB}{DE} = \frac{BC}{EF}$, and you constructed $PB = DE$. With some algebra and substitution, you can get $EF = BQ$. Then the two triangles will be congruent by the SAS Congruence Postulate.

Here is the two-column proof.

Statement	**Reason**
1. Locate P such that $PB = DE$	**1.** Segment Duplication Postulate
2. Construct $\overline{PQ} \parallel \overline{AC}$	**2.** Parallel Postulate
3. $\angle A \cong \angle QPB$	**3.** CA Postulate
4. $\angle B \cong \angle B$	**4.** Reflexive property of congruence
5. $\triangle ABC \sim \triangle PBQ$	**5.** AA Similarity Postulate
6. $\frac{AB}{PB} = \frac{BC}{BQ}$	**6.** Corresponding sides of similar triangles are proportional (CSSTP)
7. $\frac{AB}{DE} = \frac{BC}{BQ}$	**7.** Substitution
8. $\frac{AB}{DE} = \frac{BC}{EF}$	**8.** Given
9. $\frac{BC}{BQ} = \frac{BC}{EF}$	**9.** Transitive property of equality
10. $BQ = EF$	**10.** Algebra operations
11. $\angle B \cong \angle E$	**11.** Given
12. $\triangle PBQ \cong \triangle DEF$	**12.** SAS Congruence Postulate
13. $\angle QPB \cong \angle D$	**13.** CPCTC
14. $\angle A \cong \angle D$	**14.** Substitution
15. $\triangle ABC \sim \triangle DEF$	**15.** AA Similarity Postulate

This proves the SAS Similarity Conjecture.

The proof in the example above may seem complicated, but it relies on triangle congruence and triangle similarity postulates. Reading the plan again can help you follow the steps in the proof.

You can now call the SAS Similarity Conjecture the SAS Similarity Theorem and add it to your theorem list.

In the following developing proof activity you will use the SAS Similarity Theorem to prove the SSS Similarity Conjecture.

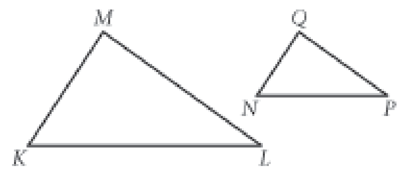
Developing Proof

Proving the SSS Similarity Conjecture

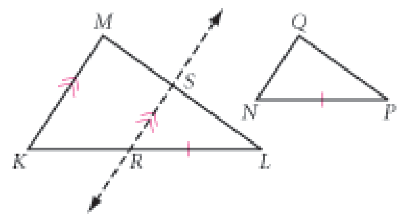
Similarity proofs can be challenging. Follow the steps and work with your group to prove the SSS Similarity Conjecture: If the three sides of one triangle are proportional to the three sides of another triangle, then the two triangles are similar.

Step 1 Identify the given and show.

Step 2 Restate what is given and what you must show in terms of this diagram.



Step 3 Plan your proof. (*Hint:* Use an auxiliary line like the one in the example.)



Step 4 Copy the first ten statements and provide the reasons. Then write the remaining steps and reasons necessary to complete the proof.

Statement	Reason
1. Locate R such that $RL = NP$	1. ? Postulate
2. Construct $\overleftrightarrow{RS} \parallel \overline{KM}$	2. ? Postulate
3. $\angle SRL \cong \angle K$	3. ? Postulate
4. $\angle RSL \cong \angle M$	4. ? Postulate
5. $\triangle KLM \sim \triangle RLS$	5. ?
6. $\frac{KL}{RL} = \frac{LM}{LS} = \frac{MK}{SR}$	6. ?
7. $\frac{KL}{NP} = \frac{LM}{LS}$	7. ?
8. $\frac{KL}{NP} = \frac{LM}{PQ}$	8. ?
9. $\frac{KL}{NP} = \frac{MK}{SR}$	9. ?
10. $\frac{KL}{NP} = \frac{MK}{QN}$	10. ?
⋮	⋮

Step 5 Draw arrows to show the flow of logic in your two-column proof.

When you have completed the proof, you can call the SSS Similarity Conjecture the SSS Similarity Theorem and add it to your theorem list.

EXERCISES

You will need

Geometry software for Exercises 17 and 20

In Exercises 1 and 2, write a proof and draw the family tree of each theorem. If the family tree is completely supported by theorems and postulates, add the theorem to your list.

1. If two triangles are similar, then corresponding altitudes are proportional to the corresponding sides. (Corresponding Altitudes Theorem)

2. If two triangles are similar, then corresponding medians are proportional to the corresponding sides. (Corresponding Medians Theorem)

In Exercises 3–10, write a proof of the conjecture. Once you have completed the proofs, add the theorems to your list. As always, you may use theorems that have been proved in previous exercises in your proofs.

3. If two triangles are similar, then corresponding angle bisectors are proportional to the corresponding sides. (Corresponding Angle Bisectors Theorem)

4. If a line passes through two sides of a triangle parallel to the third side, then it divides the two sides proportionally. (Parallel/Proportionality Theorem)

5. If a line passes through two sides of a triangle dividing them proportionally, then it is parallel to the third side. (Converse of the Parallel/Proportionality Theorem)

6. If you drop an altitude from the vertex of a right angle to its hypotenuse, then it divides the right triangle into two right triangles that are similar to each other and to the original right triangle. (Three Similar Right Triangles Theorem)

7. The length of the altitude to the hypotenuse of a right triangle is the geometric mean of the lengths of the two segments on the hypotenuse. (Altitude to the Hypotenuse Theorem)

8. The Pythagorean Theorem

9. Converse of the Pythagorean Theorem

10. If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and one leg of another right triangle, then the two right triangles are congruent. (Hypotenuse-Leg Theorem)

11. Create a family tree for the Parallel/Proportionality Theorem.

12. Create a family tree for the SSS Similarity Theorem.

13. Create a family tree for the Pythagorean Theorem.

This monument in Wellington, New Zealand, was designed by Maori architect Rewi Thompson. How would you describe the shape of the monument? How might the artist have used geometry in planning the construction?

Review

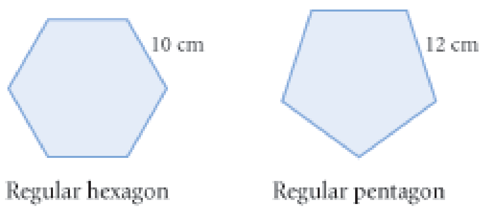
14. A circle with diameter 9.6 cm has two parallel chords with lengths 5.2 cm and 8.2 cm. How far apart are the chords? Find two possible answers.

15. Choose **A** if the value is greater in the regular hexagon.

Choose **B** if the value is greater in the regular pentagon.

Choose **C** if the values are equal in both figures.

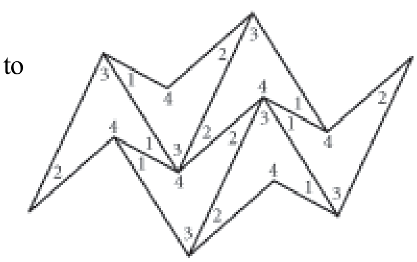
Choose **D** if it cannot be determined which value is greater.



a. Perimeter **b.** Apothem **c.** Area

d. Sum of interior angles **e.** Sum of exterior angles

16. ***Mini-Investigation*** Cut out a small nonsymmetric concave quadrilateral. Number the vertices. Use your cut-out as a template to create a tessellation. Trace about 10 images that fit together to cover part of the plane. Number the vertices of each image to match the numbers on your cut-out.



a. Draw two different translation vectors that map your tessellation onto itself. How do these two vectors relate to your original quadrilateral?

b. Pick a quadrilateral in your tessellation. What transformation will map the quadrilateral you picked onto an adjacent quadrilateral? With that transformation, what happens to the rest of the tessellation?

17. ***Technology*** Use geometry software to draw a small nonsymmetric concave quadrilateral.

a. Describe the transformations that will make the figure tessellate.

b. Describe two different translation vectors that map your tessellation onto itself.

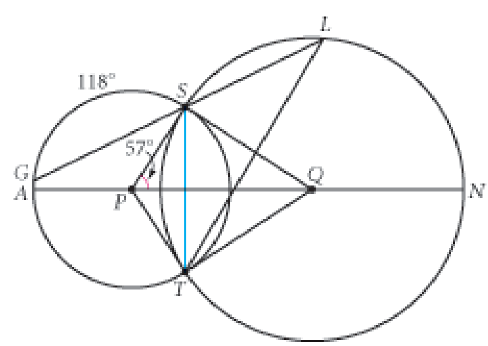
18. Given:

Circles P and Q

$\overline{PS}$ and $\overline{PT}$ are tangent to circle Q

$m\angle SPQ = 57°$

$m\overset{\frown}{GS} = 118°$



Find:

a. $m\angle GLT$ **b.** $m\angle SQT$

c. $m\angle TSQ$ **d.** $m\overset{\frown}{SL}$

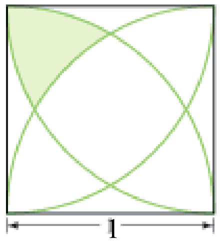
e. Explain why $PSQT$ is cyclic.

f. Explain why $\overline{SQ}$ is tangent.

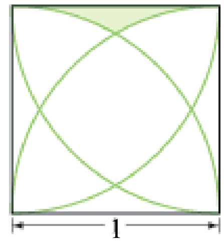
19. Each arc is a quarter of a circle with its center at a vertex of the square.

Given: The square has side length 1 unit **Find:** The shaded area

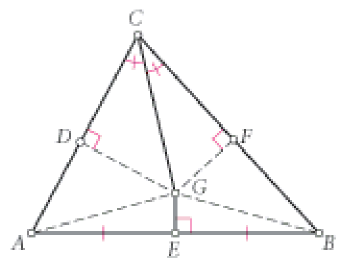
a. Shaded area $= \underline{?}$ Ⓗ



b. Shaded area $= \underline{?}$ Ⓗ



20. ***Technology*** The diagram below shows a scalene triangle with angle bisector $\overline{CG}$, and perpendicular bisector $\overline{GE}$ of side $\overline{AB}$. Study the diagram.



a. Which triangles are congruent?

b. You can use congruent triangles to prove that $\triangle ABC$ is isosceles. How?

c. Given a scalene triangle, you proved that it is isosceles. What's wrong with this proof?

d. Use geometry software to re-create the construction. What does the sketch tell you about what's wrong?

21. Dakota Davis is at an archaeological dig where he has uncovered a stone voussoir that resembles an isosceles trapezoidal prism. Each trapezoidal face has bases that measure 27 cm and 32 cm, and congruent legs that measure 32 cm each. Help Dakota determine the rise and span of the arch when it was standing, and the total number of voussoirs. Explain your method.

IMPROVING YOUR ALGEBRA SKILLS

The Eye Should Be Quicker Than the Hand

How fast can you answer these questions?

1. If $2x + y = 12$ and $3x - 2y = 17$, what is $5x - y$?
2. If $4x - 5y = 19$ and $6x + 7y = 31$, what is $10x + 2y$?
3. If $3x + 2y = 11$ and $2x + y = 7$, what is $x + y$?

Coordinate Proof

You can prove conjectures involving midpoints, slope, and distance using analytic geometry. When you do this, you create a **coordinate proof.** Coordinate proofs rely on the premises of geometry and these three properties from algebra.

Coordinate Midpoint Property

If (x_1, y_1) and (x_2, y_2) are the coordinates of the endpoints of a segment, then the coordinates of the midpoint are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Parallel Slope Property

In a coordinate plane, two distinct lines are parallel if and only if their slopes are equal.

Perpendicular Slope Property

In a coordinate plane, two nonvertical lines are perpendicular if and only if their slopes are opposite reciprocals of each other.

For coordinate proofs, you also use the coordinate version of the Pythagorean Theorem, the distance formula.

Distance Formula

The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ or $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The process you use in a coordinate proof contains the same five tasks that you learned in Lesson 13.2. However, in Task 2, you draw and label a diagram on a coordinate plane. Locate the vertices and other points of your diagram so that they reflect the given information, yet their coordinates do not restrict the generality of your diagram. In other words, do not assume any extra properties for your figure, besides the ones given in its definition.

EXAMPLE A Write a coordinate proof of the Square Diagonals Conjecture: The diagonals of a square are congruent and are perpendicular bisectors of each other.

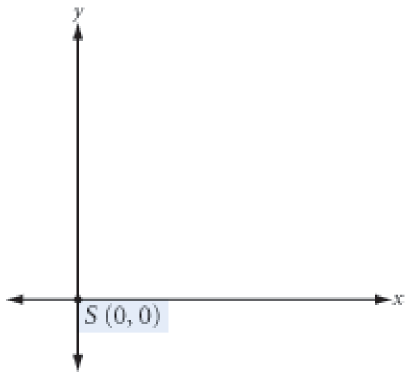
▶ ***Solution***

Task 1

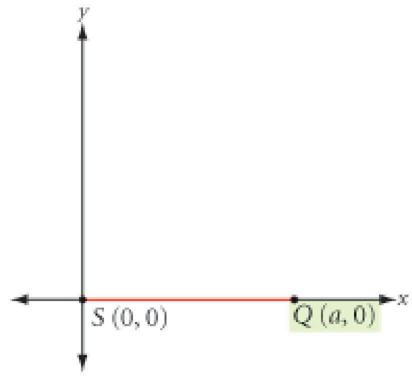
Given: A square with both diagonals

Show: The diagonals are congruent and are perpendicular bisectors of each other

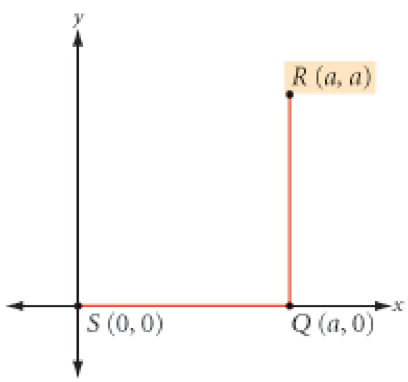
Task 2



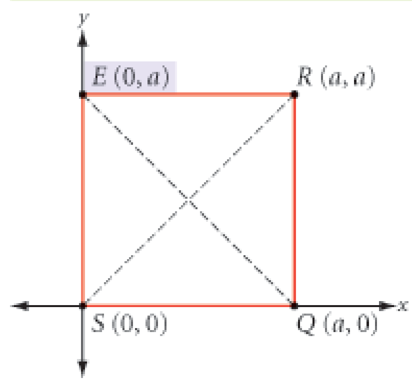
1. Placing one vertex at the origin will simplify later calculations because it is easy to work with zeros.



2. Placing the second vertex on the x-axis also simplifies calculations because the y-coordinate is zero. To remain general, call the x-coordinate a.



3. $\overline{RQ}$ needs to be vertical to form a right angle with $\overline{SQ}$, which is horizontal. $\overline{RQ}$ also needs to be the same length. So, R is placed a units vertically above Q.



4. The last vertex is placed a units above S.

You can check that $SQRE$ fits the definition of a square—an equiangular, equilateral parallelogram.

$$\text{Slope of } \overline{SQ} = \frac{0-0}{a-0} = \frac{0}{a} = 0$$

$$SQ = \sqrt{(a-0)^2 + (0-0)^2} = \sqrt{a^2} = a$$

$$\text{Slope of } \overline{QR} = \frac{a-0}{a-a} = \frac{a}{0} \text{ (undefined)}$$

$$QR = \sqrt{(a-a)^2 + (a-0)^2} = \sqrt{a^2} = a$$

Slope of $\overline{RE} = \frac{a - a}{0 - a} = \frac{0}{-a} = 0$

$RE = \sqrt{(0 - a)^2 + (a - a)^2} = \sqrt{a^2} = a$

Slope of $\overline{ES} = \frac{0 - a}{0 - 0} = \frac{-a}{0}$ (undefined)

$ES = \sqrt{(0 - 0)^2 + (0 - a)^2} = \sqrt{a^2} = a$

Opposite sides have the same slope and are therefore parallel, so $SQRE$ is a parallelogram. Also, from the slopes, $\overline{SQ}$ and $\overline{RE}$ are horizontal and $\overline{QR}$ and $\overline{ES}$ are vertical, so all angles are right angles and the parallelogram is equiangular. Lastly, all the sides have the same length, so the parallelogram is equilateral. $SQRE$ is an equiangular, equilateral parallelogram and is a square by definition.

Task 3

Given: Square $SQRE$ with diagonals $\overline{SR}$ and $\overline{QE}$

Show: $\overline{SR} \cong \overline{QE}$, $\overline{SR}$ and $\overline{QE}$ bisect each other, and $\overline{SR} \perp \overline{QE}$

Task 4

To show that $\overline{SR} \cong \overline{QE}$, you must show that both segments have the same length. To show that $\overline{SR}$ and $\overline{QE}$ bisect each other, you must show that the segments share the same midpoint. To show that $\overline{SR} \perp \overline{QE}$, you must show that the segments have opposite reciprocal slopes. Because you know the coordinates of the endpoints of both $\overline{SR}$ and $\overline{QE}$, you can use the distance formula, the coordinate midpoint property, and the definition of slope to do the necessary calculations, and to show that the perpendicular slope property is satisfied.

Task 5

Use the distance formula to find SR and QE.

$SR = \sqrt{(a - 0)^2 + (a - 0)^2} = \sqrt{2a^2} = a\sqrt{2}$

$QE = \sqrt{(a - 0)^2 + (0 - a)^2} = \sqrt{2a^2} = a\sqrt{2}$

So, by the definition of congruence, $\overline{SR} \cong \overline{QE}$ because both segments have the same length.

Use the coordinate midpoint property to find the midpoints of $\overline{SR}$ and $\overline{QE}$.

Midpoint of $\overline{SR} = \left(\frac{0 + a}{2}, \frac{0 + a}{2}\right) = (0.5a, 0.5a)$

Midpoint of $\overline{QE} = \left(\frac{0 + a}{2}, \frac{a + 0}{2}\right) = (0.5a, 0.5a)$

So, $\overline{SR}$ and $\overline{QE}$ bisect each other because both segments have the same midpoint.

Finally, compare the slopes of $\overline{SR}$ and $\overline{QE}$.

Slope of $\overline{SR} = \frac{a-0}{a-0} = 1$

Slope of $\overline{QE} = \frac{a-0}{0-a} = -1$

So, $\overline{SR} \perp \overline{QE}$ by the perpendicular slope property because the segments have opposite reciprocal slopes.

Therefore, the diagonals of a square are congruent and are perpendicular bisectors of each other.

Add the Square Diagonals Theorem to your list.

Here's another example. See if you can recognize how the five tasks result in this proof.

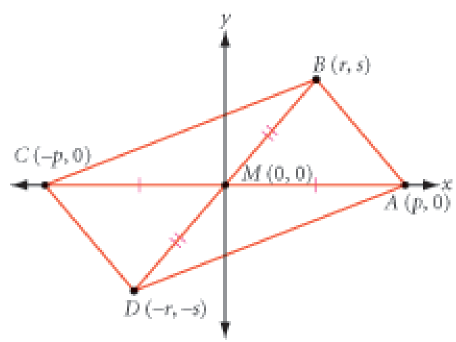
EXAMPLE B

Write a coordinate proof of this conditional statement: If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Solution

Given: Quadrilateral $ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$ that bisect each other (common midpoint M)

Show: $ABCD$ is a parallelogram



Proof

Slope of $\overline{AB} = \frac{s-0}{r-p} = \frac{s}{r-p}$

Slope of $\overline{BC} = \frac{0-s}{-p-r} = \frac{-(s)}{-(p+r)} = \frac{s}{p+r}$

Slope of $\overline{CD} = \frac{-s-0}{-r-(-p)} = \frac{-(s)}{-(r-p)} = \frac{s}{r-p}$

Slope of $\overline{DA} = \frac{0-(-s)}{p-(-r)} = \frac{s}{p+r}$

Opposite sides $\overline{AB}$ and $\overline{CD}$ have equal slopes $\frac{s}{r-p}$. Opposite sides $\overline{BC}$ and $\overline{DA}$ have equal slopes, $\frac{s}{p+r}$. So, each pair is parallel by the parallel slope property. Therefore, quadrilateral $ABCD$ is a parallelogram by definition. Add this theorem to your list.

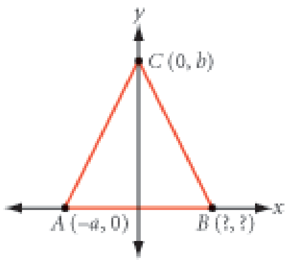
It is clear from these examples that creating a diagram on a coordinate plane is a significant challenge in a coordinate proof. The first seven exercises will give you some more practice creating these diagrams.

EXERCISES

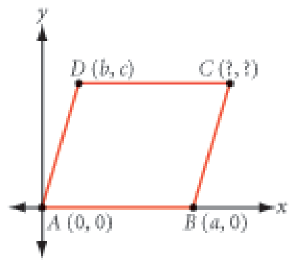
Practice Your Skills

In Exercises 1–3, each diagram shows a convenient general position of a polygon on a coordinate plane. Find the missing coordinates.

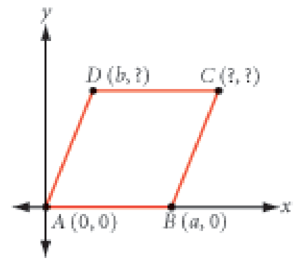
1. Triangle *ABC* is isosceles.



2. Quadrilateral *ABCD* is a parallelogram.



3. Quadrilateral *ABCD* is a rhombus.



In Exercises 4–7, draw each figure on a coordinate plane. Assign general coordinates to each point of the figure. Then use the coordinate midpoint property, parallel slope property, perpendicular slope property, and/or the distance formula to check that the coordinates you have assigned meet the definition of the figure.

4. Rectangle *RECT*

5. Triangle *TRI* with its three midsegments

6. Isosceles trapezoid *TRAP*

7. Equilateral triangle *EQU*

In Exercises 8–13, write a coordinate proof of each conjecture. If it cannot be proven, write "cannot be proven."

8. The diagonals of a rectangle are congruent.

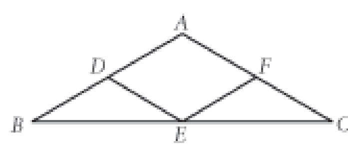
9. The midsegment of a triangle is parallel to the third side and half the length of the third side.

10. The midsegment of a trapezoid is parallel to the bases.

11. If only one diagonal of a quadrilateral is the perpendicular bisector of the other diagonal, then the quadrilateral is a kite.

12. The figure formed by connecting the midpoints of the sides of a quadrilateral is a parallelogram.

13. The quadrilateral formed by connecting the midpoint of the base to the midpoint of each leg in an isosceles triangle is a rhombus.



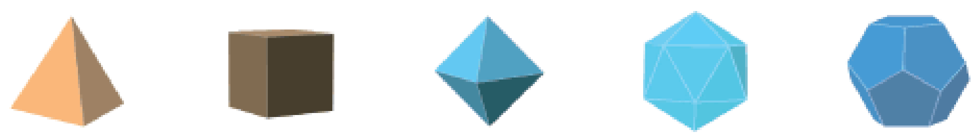
E is the midpoint of base $\overline{BC}$.
D and F are the midpoints of the legs.

project

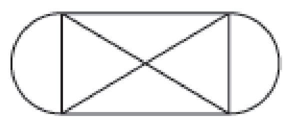
SPECIAL PROOFS OF SPECIAL CONJECTURES

In this project your task is to research and present logical arguments in support of one or more of these special properties.

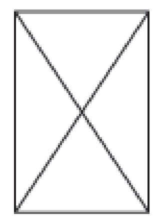
1. Prove that there are only five regular polyhedra.



2. You discovered Euler's rule for determining whether a planar network can or cannot be traveled. Write a proof defending Euler's formula for networks.

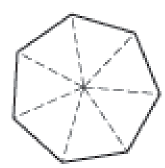


Yes

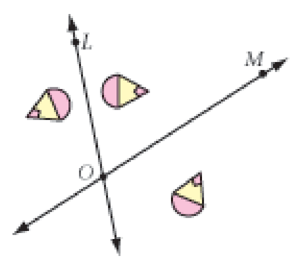


No

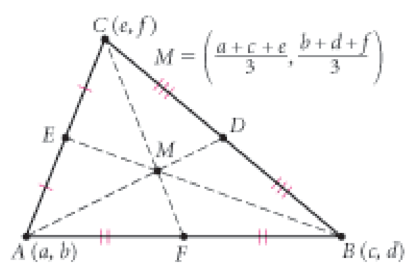
3. You discovered that the formula for the sum of the measures of the interior angles of an n-gon is $(n - 2)180°$. Prove that this formula is correct.



4. You discovered that the composition of two reflections over intersecting lines is equivalent to one rotation. Prove that this always works.



5. The coordinates of the centroid of a triangle are equal to the average of the coordinates of the triangle's three vertices. Prove that this is always true.



6. Prove that $\sqrt{2}$ is irrational.

7. When you explored all the 1-uniform tilings (Archimedean tilings), you discovered that there are exactly 11 Archimedean tilings of the plane. Prove that there are exactly 11.

Exploration

Non-Euclidean Geometries

Have you ever changed the rules of a game? Sometimes, changing just one simple rule creates a completely different game. You can compare geometry to a game whose rules are postulates. If you change even one postulate, you may create an entirely new geometry.

Hungarian mathematician János Bolyai, one of the discoverers of hyperbolic geometry, said, "I have discovered such wonderful things that I was amazed . . . out of nothing I have created a strange new universe."

Euclidean geometry—the geometry you learned in this course—is based on several postulates. A postulate, according to the contemporaries of Euclid, is an obvious truth that cannot be derived from other postulates.

The list below contains the first five of Euclid's postulates.

Postulate 1: You can draw a straight line through any two points.

Postulate 2: You can extend any segment indefinitely.

Postulate 3: You can draw a circle with any given point as center and any given radius.

Postulate 4: All right angles are equal.

Postulate 5: Through a given point not on a given line, you can draw exactly one line that is parallel to the given line.

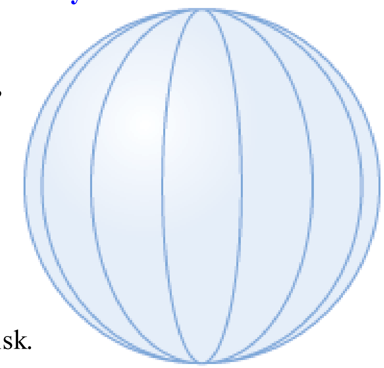
The fifth postulate, known as the Parallel Postulate, does not seem as obvious as the others. In fact, for centuries, many mathematicians did not believe it was a postulate at all and tried to show that it could be proved using the other postulates. Attempting to use indirect proof, mathematicians began by assuming that the fifth postulate was false and then tried to reach a logical contradiction.

If the Parallel Postulate is false, then one of these assumptions must be true.

Assumption 1: Through a given point not on a given line, you can draw *more than one line* parallel to the given line.

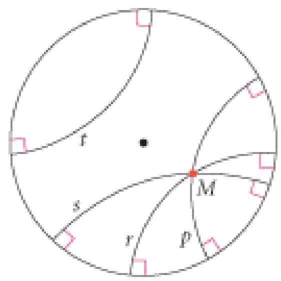
Assumption 2: Through a given point not on a given line, you can draw *no line* parallel to the given line.

Interestingly, neither of these assumptions contradict any of Euclid's other postulates. Assumption 1 leads to a new deductive system of non-Euclidean geometry, called **hyperbolic geometry.** Assumption 2 leads to another non-Euclidean system, called **elliptic geometry.**



One type of elliptic geometry called **spherical geometry** applies to lines and angles on a sphere. On Earth, if you walk in a "straight line" indefinitely, what shape will your path take? Theoretically, if you walk long enough, you will end up back at the same point, after walking a complete circle around Earth! (Find a globe and check it!) So, on a sphere, a "straight line" is not a line at all, but a circle.

Hyperbolic geometry is confined to a circular disk. The edges of the disk represent infinity so lines curve and come to an end at the edge of the circle. This may sound like a strange model, but it fits physicists' theory that we live in a closed universe.



In hyperbolic geometry, many lines can be drawn through a point parallel to another line. Lines *p*, *r*, and *s* all pass through point *M* and are parallel to line *t*.

In this activity you will explore spherical geometry.

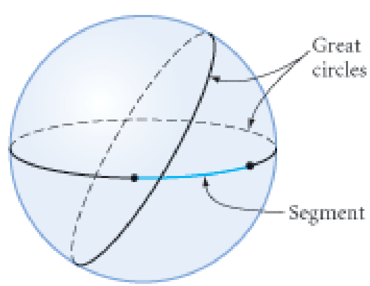
Activity

Spherical Geometry

You will need

- a sphere that you can draw on
- a compass

You can use the surface of a sphere to explore spherical geometry. Of course, you can't draw a straight line on a sphere. On a plane, the shortest distance between two points is measured along a line. On a sphere, the shortest distance between two points is measured along a great circle. Recall that a **great circle** is a circle on the surface of a sphere whose diameter passes through the center of the sphere. So, in spherical geometry, a "segment" is an arc of a great circle.



In spherical geometry, “lines” (that is, great circles) never end; however, their length is finite! Because all great circles have the same diameter, all “lines” have the same length.

Any type of elliptic geometry must satisfy the assumption that, through a given point not on a given line, there are *no* lines parallel to the given line. Simply put, there are no parallel lines in elliptic geometry. In spherical geometry, all great circles intersect, so spherical geometry satisfies this assumption.

Step 1 Write a set of postulates for spherical geometry by rewriting Euclid’s first five postulates. Replace the word *line* with the words *great circle.*

Step 2 In Euclidean geometry, two lines that are perpendicular to the same line are parallel to each other. This is not true in spherical geometry. On your sphere, draw an example of two “lines” that are perpendicular to the same “line” but that are not parallel to each other.

Step 3 On your sphere, show that two points do not always determine a unique “line.”

Step 4 Draw an isosceles triangle on your sphere. (Remember, the “segments” that form the sides of a triangle must be arcs of great circles.) Does the Isosceles Triangle Theorem appear to hold in spherical geometry?

Step 5 In spherical geometry, the sum of the measures of the three angles of a triangle is always *greater than* 180°. Draw a triangle on your model and use it to help you explain why this makes sense.

Step 6 In Euclidean geometry, no triangle can have two right angles. But in spherical geometry, a triangle can have three right angles. Find such a triangle and sketch it.

Japanese *temari* balls, colorful balls made of thread or scrap material, are embroidered with geometric designs derived from nature, like flowers or trees. Also called “princess balls,” they originated in 700 C.E., when young nobility made them from silk and gave them as gifts. Notice that each “line segment” in the design of a *temari* ball is actually an arc of a great circle.

CHAPTER 13 REVIEW

In this course you have discovered geometry properties and made conjectures based on inductive reasoning. You have also used deductive reasoning to explain why some of your conjectures were true. In this chapter you have focused on geometry as a deductive system. You learned about the premises of geometry. Starting fresh with these premises, you built a system of theorems.

By discovering geometry and then examining it as a mathematical system, you have been following in the footsteps of mathematicians throughout history. Your discoveries gave you an understanding of how geometry works. Proofs gave you the tools for taking apart your discoveries and understanding why they work.

Keymath.com Practice Your Skills

EXERCISES

In Exercises 1–7, identify each statement as true or false. For each false statement, sketch a counterexample or explain why it is false.

1. If one pair of sides of a quadrilateral are parallel and the other pair of sides are congruent, then the quadrilateral is a parallelogram.

2. If consecutive angles of a quadrilateral are supplementary, then the quadrilateral is a parallelogram.

3. If the diagonals of a quadrilateral are congruent, then the quadrilateral is a rectangle.

4. Two exterior angles of an obtuse triangle are obtuse.

5. The opposite angles of a quadrilateral inscribed within a circle are congruent.

6. The diagonals of a trapezoid bisect each other.

7. The midpoint of the hypotenuse of a right triangle is equidistant from all three vertices.

In Exercises 8–12, complete each statement.

8. A tangent is _?_ to the radius drawn to the point of tangency.

9. Tangent segments from a point to a circle are _?_.

10. The perpendicular bisector of a chord passes through _?_.

11. The three midsegments of a triangle divide the triangle into _?_.

12. A lemma is _?_.

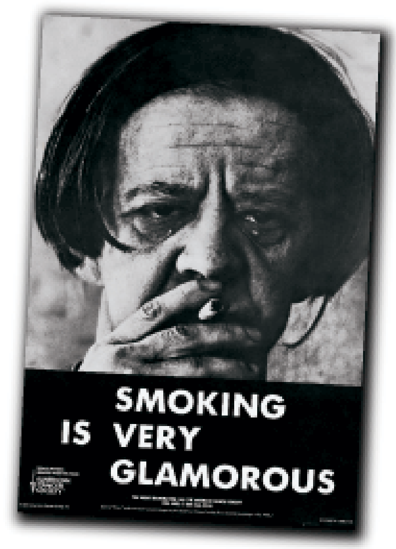
13. Restate this conjecture as a conditional: The segment joining the midpoints of the diagonals of a trapezoid is parallel to the bases.

14. Sometimes a proof requires a construction. If you need an angle bisector in a proof, which postulate allows you to construct one?

15. If an altitude is needed in a proof, which postulate allows you to construct one?

16. Describe the procedure for an indirect proof.

17. **a.** What point is this anti-smoking poster trying to make?
b. Write an indirect argument to support your answer to part a.



Developing Proof In Exercises 18–23, identify each statement as true or false. If true, prove it. If false, give a counterexample or explain why it is false.

18. If the diagonals of a parallelogram bisect the angles, then the parallelogram is a square.

19. The angle bisectors of one pair of base angles of an isosceles trapezoid are perpendicular.

20. The perpendicular bisectors to the congruent sides of an isosceles trapezoid are perpendicular.

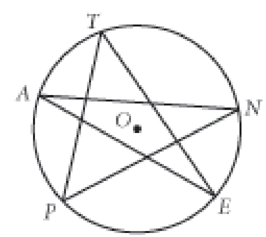
21. The segment joining the feet of the altitudes on the two congruent sides of an isosceles triangle is parallel to the third side.

22. The diagonals of a rhombus are perpendicular.

23. The bisectors of a pair of opposite angles of a parallelogram are parallel.

Developing Proof In Exercises 24–27, devise a plan and write a proof of each conjecture.

24. Refer to the figure at right.
Given: Circle O with chords $\overline{PN}$, $\overline{ET}$, $\overline{NA}$, $\overline{TP}$, $\overline{AE}$
Show: $m\angle P + m\angle E + m\angle N + m\angle T + m\angle A = 180°$



25. If a triangle is a right triangle, then it has at least one angle whose measure is less than or equal to 45°.

26. Prove the Triangle Midsegment Conjecture.

27. Prove the Trapezoid Midsegment Conjecture.

Developing Proof In Exercises 28–30, use construction tools or geometry software to perform each mini-investigation. Then make a conjecture and prove it.

28. ***Mini-Investigation*** Construct a rectangle. Construct the midpoint of each side. Connect the four midpoints to form another quadrilateral.
a. What do you observe about the quadrilateral formed? From a previous theorem, you already know that the quadrilateral is a parallelogram. State a conjecture about the type of parallelogram formed.
b. Prove your conjecture.

29. ***Mini-Investigation*** Construct a rhombus. Construct the midpoint of each side. Connect the four midpoints to form another quadrilateral.

   a. You know that the quadrilateral is a parallelogram, but what type of parallelogram is it? State a conjecture about the parallelogram formed by connecting the midpoints of a rhombus.

   b. Prove your conjecture.

30. ***Mini-Investigation*** Construct a kite. Construct the midpoint of each side. Connect the four midpoints to form another quadrilateral.

   a. State a conjecture about the parallelogram formed by connecting the midpoints of a kite.

   b. Prove your conjecture.

31. ***Developing Proof*** Prove this theorem: If two chords intersect in a circle, the product of the segment lengths on one chord is equal to the product of the segment lengths on the other chord.

Assessing What You've Learned

WRITE IN YOUR JOURNAL How does the deductive system in geometry compare to the underlying organization in your study of science, history, and language?

UPDATE YOUR PORTFOLIO Choose a project or a challenging proof you did in this chapter to add to your portfolio.

ORGANIZE YOUR NOTEBOOK Review your notebook to be sure it's complete and well organized. Be sure you have all the theorems on your theorem list. Write a one-page summary of Chapter 13.

PERFORMANCE ASSESSMENT While a classmate, friend, family member, or teacher observes, demonstrate how to prove one or more of the theorems proved in this chapter. Explain what you're doing at each step.

GIVE A PRESENTATION Give a presentation on a puzzle, exercise, or project from this chapter. Work with your group, or try presenting on your own.