



*Perhaps all I pursue is
astonishment and so I try to
awaken only astonishment in my
viewers. Sometimes "beauty" is a
nasty business.*

M. C. ESCHER

Verblifa tin, M. C. Escher, 1963
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OBJECTIVES

In this chapter you will

- explore and define many three-dimensional solids
- discover formulas for finding the volumes of prisms, pyramids, cylinders, cones, and spheres
- learn how density is related to volume
- derive a formula for the surface area of a sphere

*Everything in nature
adheres to the cone,
the cylinder, and the cube.*

PAUL CEZANNE

The Geometry of Solids

Most of the geometric figures you have worked with so far have been flat plane figures with two dimensions—base and height. In this chapter you will work with solid figures with three dimensions—length, width, and height. Most real-world solids, like rocks and plants, are very irregular, but many others are geometric. Some real-world geometric solids occur in nature: viruses, oranges, crystals, the earth itself. Others are human-made: books, buildings, baseballs, soup cans, ice cream cones.



This amethyst crystal is an irregular solid, but parts of it have familiar shapes.

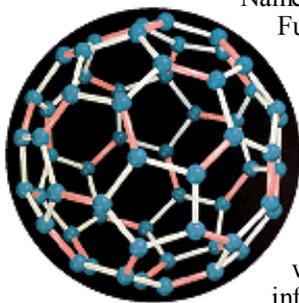


Science

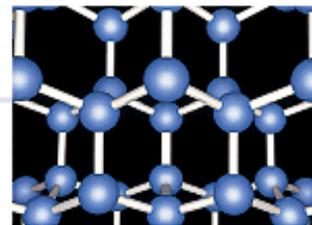
CONNECTION

Three-dimensional geometry plays an important role in the structure of molecules. For example, when carbon atoms are arranged in a very rigid network, they form diamonds, one of the earth's hardest materials. But when carbon atoms are arranged in planes of hexagonal rings, they form graphite, a soft material used in pencil lead.

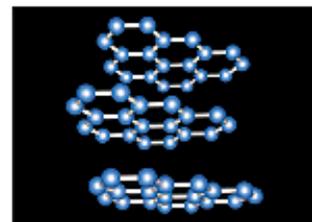
Carbon atoms can also bond into very large molecules. Named fullerenes, after U.S. engineer Buckminster Fuller (1895–1983), these carbon molecules have the same symmetry as a soccer ball, as shown at left. They are popularly called buckyballs.



Researchers have recently discovered a similar phenomenon with gold atoms. Gold clusters of fewer than 16 atoms are flat, and clusters of more than 18 atoms form a pyramid, but clusters of 16, 17, or 18 form a hollow cage. Like buckyballs, these hollow gold cages may be able to house other atoms, which may have applications in nanotechnology and medicine. For more information, visit www.keymath.com/DG.

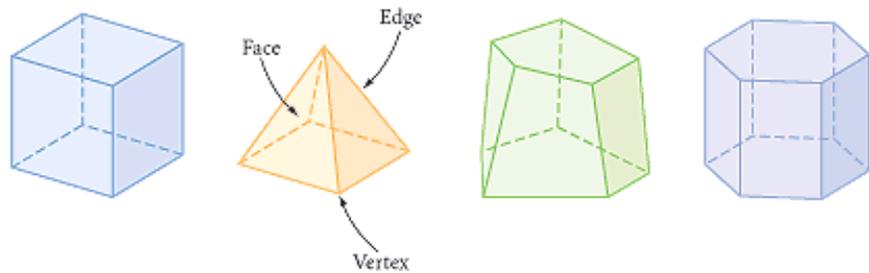


The geometry of diamonds

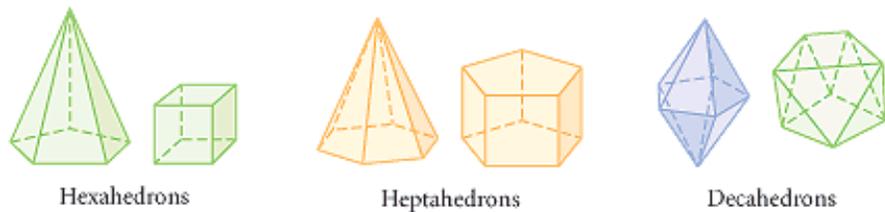


The geometry of graphite

A solid formed by polygons that enclose a single region of space is called a **polyhedron**. The flat polygonal surfaces of a polyhedron are called its **faces**. Although a face of a polyhedron includes the polygon and its interior region, we identify the face by naming the polygon that encloses it. A segment where two faces intersect is called an **edge**. The point of intersection of three or more edges is called a **vertex** of the polyhedron.



Just as a polygon is classified by its number of sides, a polyhedron is classified by its number of faces. The prefixes for polyhedrons are the same as they are for polygons with one exception: A polyhedron with four faces is called a **tetrahedron**. Here are some examples of polyhedrons.



If each face of a polyhedron is enclosed by a regular polygon, and each face is congruent to the other faces, and the faces meet at each vertex in exactly the same way, then the polyhedron is called a **regular polyhedron**. The regular polyhedron shown at right is called a regular dodecahedron because it has 12 faces.

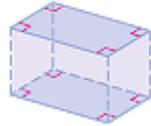


The Ramat Polin housing complex in Jerusalem, Israel, has many polyhedral shapes.



A **prism** is a special type of polyhedron, with two faces called **bases**, that are congruent, parallel polygons. The other faces of the polyhedron, called **lateral faces**, are parallelograms that connect the corresponding sides of the bases.

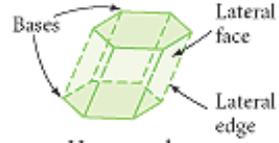
The lateral faces meet to form the **lateral edges**. Each solid shown below is a prism.



Rectangular prism



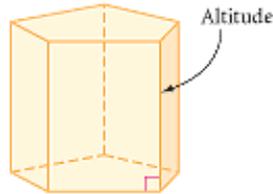
Triangular prism



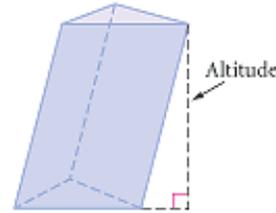
Hexagonal prism

Prisms are classified by their bases. For example, a prism with triangular bases is a triangular prism, and a prism with hexagonal bases is a hexagonal prism.

A prism whose lateral faces are rectangles is called a **right prism**. Its lateral edges are perpendicular to its bases. A prism that is not a right prism is called an **oblique prism**. The **altitude** of a prism is any perpendicular segment from one base to the plane of the other base. The length of an altitude is the **height** of the prism.



Right pentagonal prism

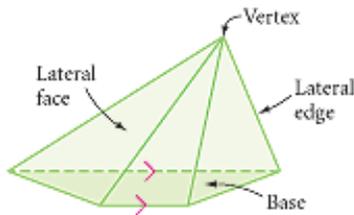


Oblique triangular prism

A **pyramid** is another special type of polyhedron. Pyramids have only one base. As in a prism, the other faces are called the lateral faces, and they meet to form the lateral edges. The common vertex of the lateral faces is the vertex of the pyramid.



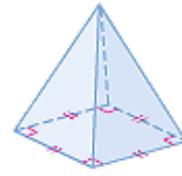
Triangular pyramid



Trapezoidal pyramid



Hexagonal pyramid



Square pyramid

Like prisms, pyramids are also classified by their bases. The pyramids of Egypt are square pyramids because they have square bases.

The altitude of the pyramid is the perpendicular segment from its vertex to the plane of its base. The length of the altitude is the height of the pyramid.



Polyhedrons are geometric solids with flat surfaces. There are also geometric solids that have curved surfaces.

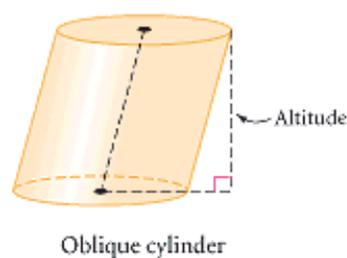
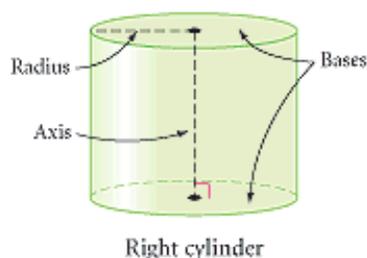
One solid with a curved surface is a **cylinder**. Soup cans, compact discs (CDs), and plumbing pipes are shaped like cylinders. Like a prism, a cylinder has two bases that are both parallel and congruent. Instead of polygons, however, the bases of cylinders are circles and their interiors. The segment connecting the centers of the bases is called the **axis** of the cylinder. The **radius** of the cylinder is the radius of a base.

If the axis of a cylinder is perpendicular to the bases, then the cylinder is a **right cylinder**. A cylinder that is not a right cylinder is an **oblique cylinder**.

The altitude of a cylinder is any perpendicular segment from the plane of one base to the plane of the other. The height of a cylinder is the length of an altitude.

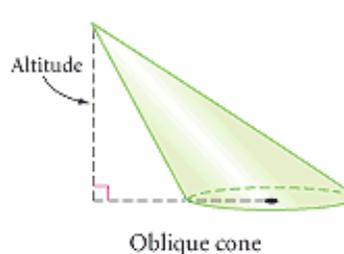
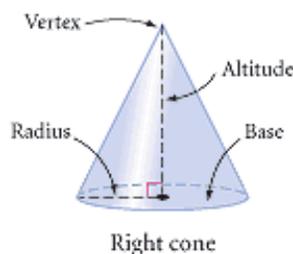


100 Cans (1962 oil on canvas), by pop art artist Andy Warhol (1925–1987), repeatedly uses the cylindrical shape of a soup can to make an artistic statement with a popular image.



Another type of solid with a curved surface is a **cone**. Funnels and ice cream cones are shaped like cones. Like a pyramid, a cone has a base and a vertex.

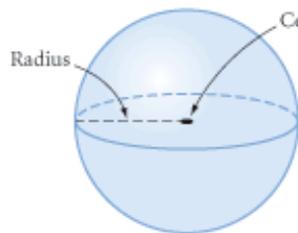
The base of a cone is a circle and its interior. The radius of a cone is the radius of the base. The vertex of a cone is the point that is the greatest perpendicular distance from the base. The altitude of a cone is the perpendicular segment from the vertex to the plane of the base. The length of the altitude is the height of a cone. If the line segment connecting the vertex of a cone with the center of its base is perpendicular to the base, then the cone is a **right cone**.



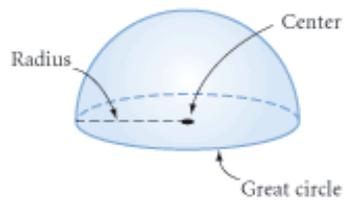
A third type of solid with a curved surface is a **sphere**. Basketballs, globes, and oranges are shaped like spheres. A sphere is the set of all points in space at a given distance from a given point. You can think of a sphere as a three-dimensional circle.



The given distance is called the **radius** of the sphere, and the given point is the **center** of the sphere. A **hemisphere** is half a sphere and its circular base. The circle that encloses the base of a hemisphere is called a **great circle** of the sphere. Every plane that passes through the center of a sphere determines a great circle. All the longitude lines on a globe of Earth are great circles. The equator is the only latitude line that is a great circle.



Sphere



Hemisphere



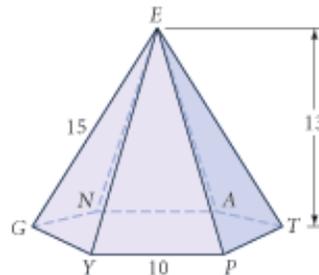
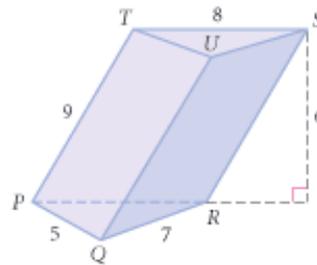
EXERCISES

1. Complete this definition:

A pyramid is a ? with one ? face (called the base) and whose other faces (lateral faces) are ? formed by segments connecting the vertices of the base to a common point (the vertex) not on the base.

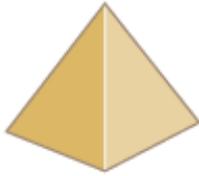
For Exercises 2–9, refer to the figures below. All measurements are in centimeters.

- Name the bases of the prism.
- Name all the lateral faces of the prism.
- Name all the lateral edges of the prism.
- What is the height of the prism?
- Name the base of the pyramid.
- Name the vertex of the pyramid.
- Name all the lateral edges of the pyramid.
- What is the height of the pyramid?

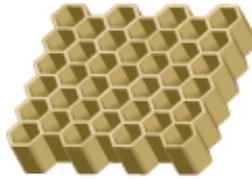


For Exercises 10–22, match each real object with a geometry term. You may use a geometry term more than once or not at all.

10. Tomb of Egyptian rulers



11. Honeycomb



- A. Cylinder
- B. Cone
- C. Square prism
- D. Square pyramid
- E. Sphere
- F. Triangular pyramid
- G. Octagonal prism
- H. Triangular prism
- I. Trapezoidal prism
- J. Rectangular prism
- K. Heptagonal pyramid
- L. Hexagonal prism
- M. Hemisphere

12. Die

13. Stop sign

14. Holder for a scoop of ice cream

15. Wedge or doorstep



16. Moon

17. Can of tuna fish

18. Box of breakfast cereal

19. Book

20. Plastic bowl with lid

21. Pup tent



22. Ingot of silver



For Exercises 23–26, draw and label each solid. Use dashed lines to show the hidden edges.

23. A triangular pyramid whose base is an equilateral triangular region (Use the proper marks to show that the base is equilateral.)

24. A hexahedron with two trapezoidal faces

25. A cylinder with a height that is twice the diameter of the base (Use x and $2x$ to indicate the height and the diameter.)

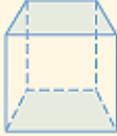
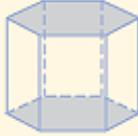
26. A right cone with a height that is half the diameter of the base

For Exercises 27–35, identify each statement as true or false. Sketch a counterexample for each false statement or explain why it is false.

27. A lateral face of a pyramid is always a triangular region.

28. A lateral edge of a pyramid is always perpendicular to the base.

29. Every slice of a prism cut parallel to the bases is congruent to the bases. 
30. When the lateral surface of a right cylinder is unwrapped and laid flat, it is a rectangle.
31. When the lateral surface of a right circular cone is unwrapped and laid flat, it is a triangle. 
32. Every section of a cylinder, parallel to the base, is congruent to the base.
33. The length of a segment from the vertex of a cone to the circular base is the height of the cone.
34. The length of the axis of a right cylinder is the height of the cylinder.
35. All slices of a sphere passing through the sphere's center are congruent.
36. **Mini-Investigation** An **antiprism** is a polyhedron with two congruent bases and lateral faces that are triangles. Complete the tables below for prisms and antiprisms. Describe any relationships you see between the number of lateral faces, total faces, edges, and vertices of related prisms and antiprisms.

	Triangular prism 	Rectangular prism 	Pentagonal prism 	Hexagonal prism 		n -gonal prism
Lateral faces	3				...	
Total faces		6			...	
Edges				18	...	
Vertices			10		...	

	Triangular antiprism 	Rectangular antiprism 	Pentagonal antiprism 	Hexagonal antiprism 		n -gonal antiprism
Lateral faces	6				...	
Total faces		10			...	
Edges				24	...	
Vertices			10		...	

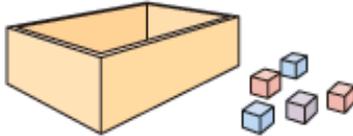
37. Write a paragraph describing the visual tricks that Belgian artist René Magritte (1898–1967) plays in his painting at right.



The Promenades of Euclid (1935 oil on canvas), René Magritte

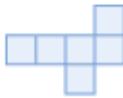
Review

For Exercises 38 and 39, how many cubes measuring 1 cm on each edge will fit into the container?

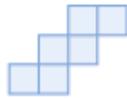


38. A box measuring 2 cm on each inside edge
39. A box measuring 3 cm by 4 cm by 5 cm on the inside edges
40. What is the maximum number of boxes measuring 1 cm by 1 cm by 2 cm that can fit within a box whose inside dimensions are 3 cm by 4 cm by 5 cm?
41. For each net, decide whether it folds to make a box. If it does, copy the net and mark each pair of opposite faces with the same symbol.

a.



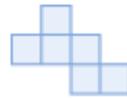
b.



c.



d.



IMPROVING YOUR VISUAL THINKING SKILLS

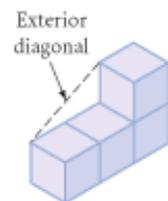
Piet Hein's Puzzle

In 1936, while listening to a lecture on quantum physics, the Danish mathematician Piet Hein (1905–1996) devised the following visual thinking puzzle:

What are all the possible nonconvex solids that can be created by joining four or fewer cubes face-to-face?

A nonconvex polyhedron is a solid that has at least one diagonal that is exterior to the solid. For example, four cubes in a row, joined face-to-face, form a convex polyhedron. But four cubes joined face-to-face into an L-shape form a nonconvex polyhedron.

Use isometric dot paper to sketch the nonconvex solids that solve Piet Hein's puzzle.



Exploration

Euler's Formula for Polyhedrons

In this activity you will discover a relationship among the vertices, edges, and faces of a polyhedron. This relationship is called Euler's Formula for Polyhedrons, named after Leonhard Euler. Let's first build some of the polyhedrons you learned about in Lesson 10.1.



Activity Toothpick Polyhedrons

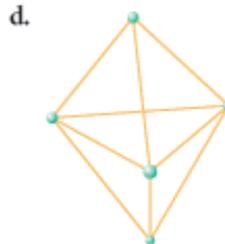
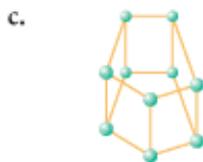
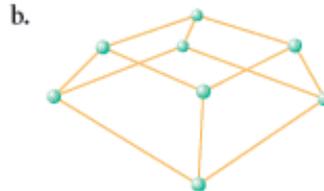
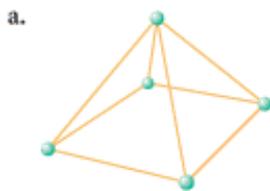
You will need

- toothpicks
- modeling clay, gumdrops, or dried peas

First, you'll model polyhedrons using toothpicks as edges and using small balls of clay, gumdrops, or dried peas as connectors.

Step 1

Build and save the polyhedrons shown in parts a–d below and described in parts e–i on the top of page 531. You may have to cut or break some sticks. Share the tasks among the group.



- e. Build a tetrahedron.
 - f. Build an octahedron.
 - g. Build a nonahedron.
 - h. Build at least two different-shaped decahedrons.
 - i. Build at least two different-shaped dodecahedrons.
- Step 2 Classify all the different polyhedrons your class built as prisms, pyramids, regular polyhedrons, or just polyhedrons.

Next, you'll look for a relationship among the vertices, faces, and edges of the polyhedrons.

- Step 3 Count the number of vertices (V), edges (E), and faces (F) of each polyhedron model. Copy and complete a chart like this one.

Polyhedron	Vertices (V)	Faces (F)	Edges (E)
⋮	⋮	⋮	⋮

- Step 4 Look for patterns in the table. By adding, subtracting, or multiplying V , F , and E (or a combination of two or three of these operations), you can discover a formula that is commonly known as Euler's Formula for Polyhedrons.

- Step 5 Now that you have discovered the formula relating the number of vertices, edges, and faces of a polyhedron, use it to answer each of these questions.
- a. Which polyhedron has 4 vertices and 6 edges? Can you build another polyhedron with a different number of faces that also has 4 vertices and 6 edges?
 - b. Which polyhedron has 6 vertices and 12 edges? Can you build another polyhedron with a different number of faces that also has 6 vertices and 12 edges?
 - c. If a solid has 8 faces and 12 vertices, how many edges will it have?
 - d. If a solid has 7 faces and 12 edges, how many vertices will it have?
 - e. If a solid has 6 faces, what are all the possible combinations of vertices and edges it can have?

Volume of Prisms and Cylinders

How much deeper would oceans be if sponges didn't live there?

STEVEN WRIGHT

In real life you encounter many volume problems. For example, when you shop for groceries, it's a good idea to compare the volumes and the prices of different items to find the best buy. When you fill a car's gas tank or when you fit last night's leftovers into a freezer dish, you fill the volume of an empty container.

Many occupations also require familiarity with volume. An engineer must calculate the volume and the weight of sections of a bridge to avoid putting too much stress on any one section. Chemists, biologists, physicists, and geologists must all make careful volume measurements in their research. Carpenters, plumbers, and painters also know and use volume relationships. A chef must measure the correct volume of each ingredient in a cake to ensure a tasty success.



A father and son test the quality of the grain harvested on their family farm. The payment they receive from the processing plant is based on the volume of grain, not its weight.



Volume is the measure of the amount of space contained in a solid. You use cubic units to measure volume: cubic inches (in^3), cubic feet (ft^3), cubic yards (yd^3), cubic centimeters (cm^3), cubic meters (m^3), and so on.

The volume of an object is the number of unit cubes that completely fill the space within the object.

$\underline{1}$
Length: 1 unit

$\begin{matrix} & 1 & \\ & \square & \\ 1 & & \end{matrix}$
Volume: 1 cubic unit

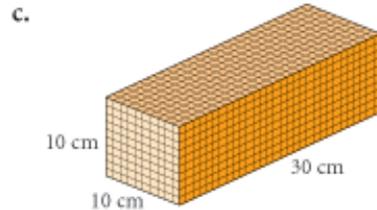
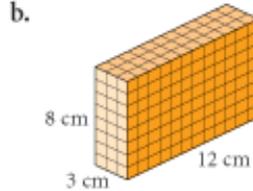
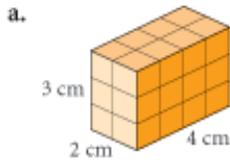
$\begin{matrix} & & & & \\ & \square & \square & \square & \\ & & & & \\ & & & & \\ & & & & \end{matrix}$
Volume: 20 cubic units



Investigation

The Volume Formula for Prisms and Cylinders

- Step 1 | Find the volume of each right rectangular prism below in cubic centimeters. That is, how many cubes measuring 1 cm on each edge will fit into each solid? Within your group, discuss different strategies for finding each volume. How could you find the volume of any right rectangular prism?



Notice that the number of cubes resting on the base equals the number of square units in the area of the base. The number of layers of cubes equals the number of units in the height of the prism. So you can use the area of the base and the height of the prism to calculate the volume.

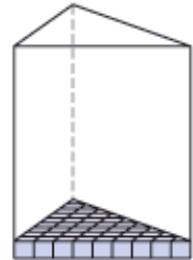
- Step 2 | Complete the conjecture.

Rectangular Prism Volume Conjecture

C-86a

If B is the area of the base of a right rectangular prism and H is the height of the solid, then the formula for the volume is $V = ?$.

In Chapter 8, you discovered that you can reshape parallelograms, triangles, trapezoids, and circles into rectangles to find their area. You can use the same method to find the areas of bases that have these shapes. Then you can multiply the area of the base by the height of the prism to find its volume. For example, to find the volume of a right triangular prism, find the area of the triangular base (the number of cubes resting on the base) and multiply it by the height (the number of layers of cubes).



So, you can extend the Rectangular Prism Volume Conjecture to all right prisms and right cylinders.

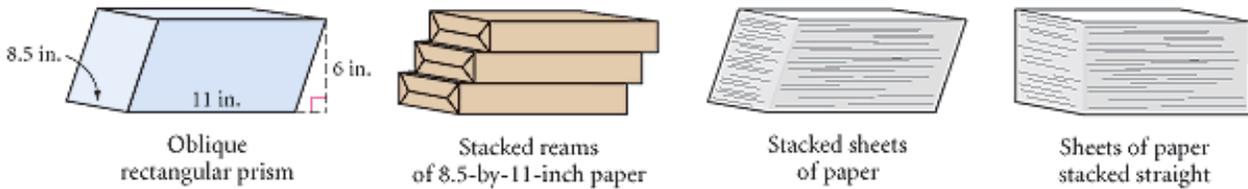
- Step 3 | Complete the conjecture.

Right Prism-Cylinder Volume Conjecture

C-86b

If B is the area of the base of a right prism (or cylinder) and H is the height of the solid, then the formula for the volume is $V = ?$.

What about the volume of an oblique prism or cylinder? You can approximate the shape of this oblique rectangular prism with a staggered stack of three reams of 8.5-by-11-inch paper. If you nudge the individual pieces of paper into a slanted stack, then your approximation can be even better.



Rearranging the paper into a right rectangular prism changes the shape, but certainly the volume of paper hasn't changed. The area of the base, 8.5 by 11 inches, didn't change and the height, 6 inches, didn't change, either.

In the same way, you can use crackers, CDs, or coins to show that an oblique cylinder has the same volume as a right cylinder with the same base and height.



Step 4 | Use the stacking model to extend the last conjecture to oblique prisms and cylinders. Complete the conjecture.

Oblique Prism-Cylinder Volume Conjecture C-86c

The volume of an oblique prism (or cylinder) is the same as the volume of a right prism (or cylinder) that has the same ? and the same ?.

Finally, you can combine the last three conjectures into one conjecture for finding the volume of any prism or cylinder, whether it's right or oblique.

Step 5 | Copy and complete the conjecture.

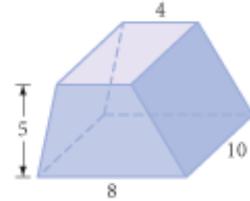
Prism-Cylinder Volume Conjecture C-86

The volume of a prism or a cylinder is the ? multiplied by the ?.

If you successfully completed the investigation, you saw that the same volume formula applies to all prisms and cylinders, regardless of the shapes of their bases. To calculate the volume of a prism or cylinder, first calculate the area of the base using the formula appropriate to its shape. Then multiply the area of the base by the height of the solid. In oblique prisms and cylinders, the lateral edges are no longer at right angles to the bases, so you do *not* use the length of the lateral edge as the height.

EXAMPLE A

Find the volume of a right trapezoidal prism that has a height of 10 cm. The two bases of the trapezoid measure 4 cm and 8 cm, and its height is 5 cm.

**► Solution**

Find the area of the base.

$$B = \frac{1}{2}h(b_1 + b_2)$$

The base is a trapezoid, so use this formula to find the area of the base.

$$B = \frac{1}{2} (5)(4 + 8) = 30$$

Substitute the given values into the equation, then simplify.

Find the volume.

$$V = BH$$

The volume of a prism is equal to the area of its base multiplied by its height.

$$V = (30)(10) = 300$$

Substitute the calculated area and given height into the equation, then simplify.

The volume is 300 cm³.

EXAMPLE B

Find the volume of an oblique cylinder that has a base with a radius of 6 inches and a height of 7 inches.

**► Solution**

Find the area of the base.

$$B = \pi r^2$$

The base is a circle, so use this formula to find the area of the base.

$$B = \pi(6)^2 = 36\pi$$

Substitute the given values into the equation, then simplify.

Find the volume.

$$V = BH$$

The volume of a prism is equal to the area of its base multiplied by its height.

$$V = (36\pi) (7) = 252\pi$$

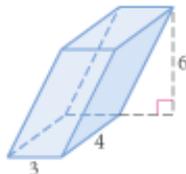
Substitute the calculated area and given height into the equation, then simplify.

The volume is 252π in³, or about 791.68 in³.

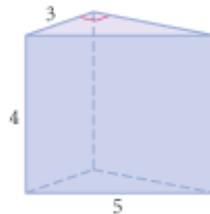
**EXERCISES**

Find the volume of each solid in Exercises 1–6. All measurements are in centimeters. Round approximate answers to the nearest hundredths.

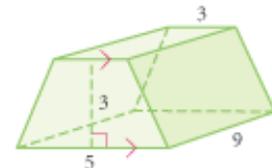
1. Oblique rectangular prism



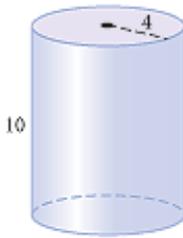
2. Right triangular prism



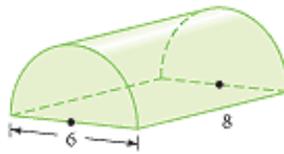
3. Right trapezoidal prism



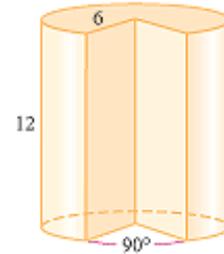
4. Right cylinder



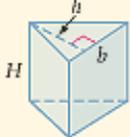
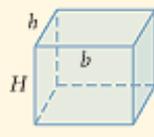
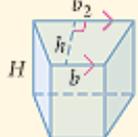
5. Right semicircular cylinder h



6. Right cylinder with a 90° slice removed h



7. Use the information about the base and height of each solid to find the volume. All measurements are given in centimeters.

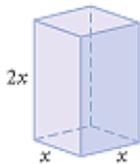
Information about base of solid	Height of solid	Right triangular prism 	Right rectangular prism 	Right trapezoidal prism 	Right cylinder 
$b = 6, b_2 = 7, h = 8, r = 3$	$H = 20$	a. $V = \text{h}$	d. $V =$	g. $V =$	j. $V =$
$b = 9, b_2 = 12, h = 12, r = 6$	$H = 20$	b. $V =$	e. $V =$	h. $V =$	k. $V =$
$b = 8, b_2 = 19, h = 18, r = 8$	$H = 23$	c. $V =$	f. $V =$	i. $V =$	l. $V =$

For Exercises 8–9, sketch and label each solid described, then find the volume.

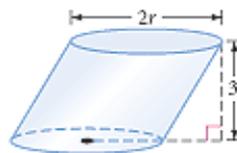
8. An oblique trapezoidal prism. The trapezoidal base has a height of 4 in. and bases that measure 8 in. and 12 in. The height of the prism is 24 in.
9. A right circular cylinder with a height of T . The radius of the base is \sqrt{Q} . h
10. Sketch and label two different rectangular prisms, each with a volume of 288 cm^3 .

In Exercises 11–13, express the volume of each solid with the help of algebra.

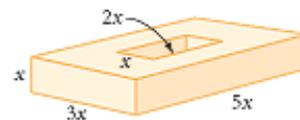
11. Right rectangular prism



12. Oblique cylinder



13. Right rectangular prism with a rectangular hole

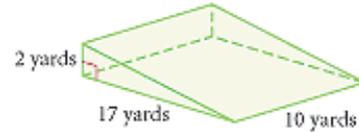


14. **Application** A cord of firewood is 128 cubic feet. Margaretta has three storage boxes for firewood that each measure 2 feet by 3 feet by 4 feet. Does she have enough space to order a full cord of firewood? A half cord? A quarter cord? Explain.

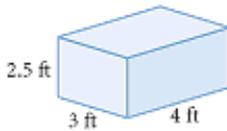
Career CONNECTION

In construction and landscaping, sand, rocks, gravel, and fill dirt are often sold by the “yard,” which actually means a cubic yard.

15. **Application** A contractor needs to build a ramp, as shown at right, from the street to the front of a garage door. How many cubic yards of fill will she need?



16. If an average rectangular block of limestone used to build the Great Pyramid of Khufu at Giza is approximately 2.5 feet by 3 feet by 4 feet, and limestone weighs approximately 170 pounds per cubic foot, what is the weight of one of the nearly 2,300,000 limestone blocks used to build the pyramid?



17. Although the Exxon *Valdez* oil spill (11 million gallons of oil) is one of the most notorious oil spills, it was small compared to the 250 million gallons of crude oil that were spilled during the 1991 Persian Gulf War. A gallon occupies 0.13368 cubic foot. How many rectangular swimming pools, each 20 feet by 30 feet by 5 feet, could be filled with 250 million gallons of crude oil?



The Great Pyramid of Khufu at Giza, Egypt, was built around 2500 B.C.E.

18. When folded, a 12-by-12-foot section of the AIDS Memorial Quilt requires about 1 cubic foot of storage. In 1996, the quilt consisted of 32,000 3-by-6-foot panels. What was the quilt’s volume in 1996? If the storage facility had a floor area of 1,500 square feet, how high did the quilt panels need to be stacked?



The NAMES Project AIDS Memorial Quilt memorializes persons all around the world who have died of AIDS. In 1996, the 32,000 panels represented less than 10% of the AIDS deaths in the United States alone, yet the quilt could cover about 19 football fields.

Review

For Exercises 19 and 20, draw and label each solid. Use dashed lines to show the hidden edges.

19. An octahedron with all triangular faces and another octahedron with at least one nontriangular face
20. A cylinder with both radius and height r , a cone with both radius and height r resting flush on one base of the cylinder, and a hemisphere with radius r resting flush on the other base of the cylinder

For Exercises 21 and 22, identify each statement as true or false. Sketch a counterexample for each false statement or explain why it is false.

21. A prism always has an even number of vertices.
22. A section of a cube is either a square or a rectangle.
23. The tower below is an unusual shape. It's neither a cylinder nor a cone. Sketch a two-dimensional figure and an axis such that if you spin your figure about the axis, it will create a solid of revolution shaped like the tower.



The Sugong Tower Mosque in Turpan, China

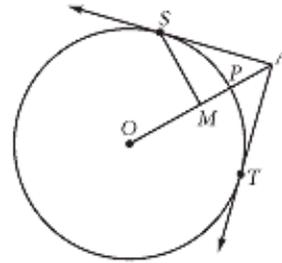
24. Do research to find a photo or drawing of a chemical model of a crystal. Sketch it. What type of polyhedral structure does it exhibit? You will find helpful Internet links at www.keymath.com/DG.

Science CONNECTION

Ice is a well-known crystal structure. If ice were denser than water, it would sink to the bottom of the ocean, away from heat sources. Eventually the oceans would fill from the bottom up with ice, and we would have an ice planet. What a cold thought!



25. Six points are equally spaced around a circular track with a 20 m radius. Ben runs around the track from one point, past the second, to the third. Al runs straight from the first point to the second, and then straight to the third. How much farther does Ben run than Al?

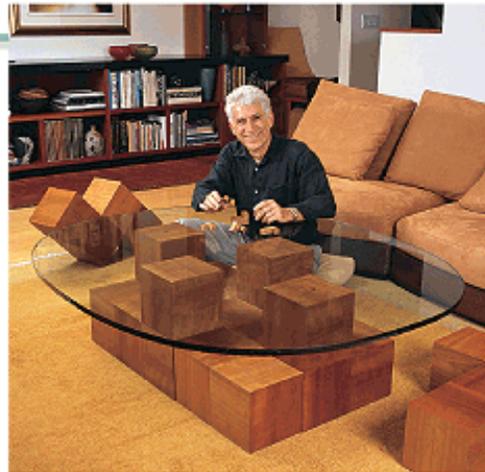


26. \overline{AS} and \overline{AT} are tangent to circle O at S and T , respectively. $m\angle SMO = 90^\circ$, $m\angle SAT = 90^\circ$, $SM = 6$. Find the exact value of PA . 

project

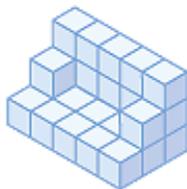
THE SOMA CUBE

If you solved Piet Hein's puzzle at the end of the previous lesson, you now have sketches of the seven nonconvex polyhedrons that can be assembled using four or fewer cubes. These seven polyhedrons consist of a total of 27 cubes: 6 sets of 4 cubes and 1 set of 3 cubes. These pieces can be arranged to form a 3-by-3-by-3 cube. The puzzle of how to put them together in a perfect cube is known as the Soma Cube puzzle. Go to www.keymath.com/DG to learn more about the Soma Cube.

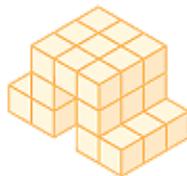


Use cubes (wood, plastic, or sugar cubes) to build one set of the seven pieces of the Soma Cube. Use glue, tape, or putty to connect the cubes.

Solve the Soma Cube puzzle: Put the pieces together to make a 3-by-3-by-3 cube. Then build these other shapes. How do you build the sofa? The tunnel? The castle? The aircraft carrier? Finally, create a shape of your own that uses all the pieces.



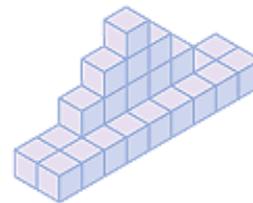
Sofa



Tunnel



Castle



Aircraft Carrier

Your project should include

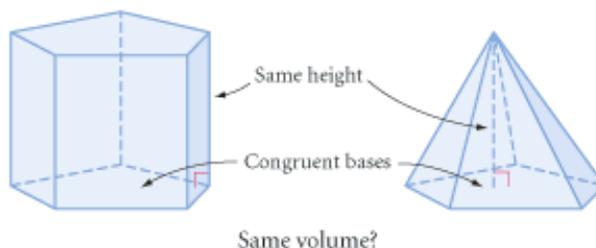
- ▶ The seven unique pieces of the Soma Cube.
- ▶ Solution to the Soma Cube puzzle and at least two of the other puzzles.
- ▶ An isometric drawing of your own shape that uses all seven pieces.

Volume of Pyramids and Cones

If I had influence with the good fairy ... I should ask that her gift to each child in the world be a sense of wonder so indestructible that it would last throughout life.

RACHEL CARSON

There is a simple relationship between the volumes of prisms and pyramids with congruent bases and the same height, and between cylinders and cones with congruent bases and the same height. You'll discover this relationship in the investigation.



Investigation

The Volume Formula for Pyramids and Cones

You will need

- container pairs of prisms and pyramids
- container pairs of cylinders and cones
- sand, rice, birdseed, or water



- Step 1 Choose a prism and a pyramid that have congruent bases and the same height.
- Step 2 Fill the pyramid, then pour the contents into the prism. About what fraction of the prism is filled by the volume of one pyramid?
- Step 3 Check your answer by repeating Step 2 until the prism is filled.
- Step 4 Choose a cone and a cylinder that have congruent bases and the same height and repeat Steps 2 and 3.
- Step 5 Compare your results with the results of others. Did you get similar results with both your pyramid-prism pair and the cone-cylinder pair? You should be ready to make a conjecture.

Pyramid-Cone Volume Conjecture

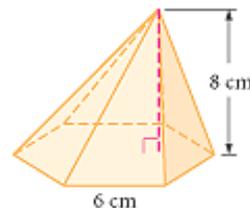
C-87

If B is the area of the base of a pyramid or a cone and H is the height of the solid, then the formula for the volume is $V = \frac{1}{3}BH$.

If you successfully completed the investigation, you probably noticed that the volume formula is the same for all pyramids and cones, regardless of the type of base they have. To calculate the volume of a pyramid or cone, first find the area of its base. Then find the product of the fraction you discovered in the investigation, the area of the base, and the height of the solid.

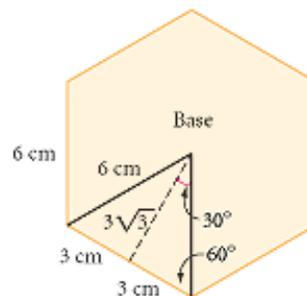
EXAMPLE A

Find the volume of a regular hexagonal pyramid with a height of 8 cm. Each side of its base is 6 cm.



► **Solution**

First, find the area the base. To find the area of a regular hexagon, you need the apothem. By the 30°-60°-90° Triangle Conjecture, the apothem is $3\sqrt{3}$ cm.



$$B = \frac{1}{2}ap$$

The area of a regular polygon is one-half the apothem times the perimeter.

$$B = \frac{1}{2}(3\sqrt{3})(36)$$

Substitute $3\sqrt{3}$ for a and 36 for p .

$$B = 54\sqrt{3}$$

Multiply.

The base has an area $54\sqrt{3}$ cm². Now find the volume of the pyramid.

$$V = \frac{1}{3}BH$$

The volume of a pyramid is one-third the area of the base times the height.

$$V = \frac{1}{3}(54\sqrt{3})(8)$$

Substitute $54\sqrt{3}$ for B and 8 for H .

$$V = 144\sqrt{3}$$

Multiply.

The volume is $144\sqrt{3}$ cm³, or approximately 249.4 cm³.

EXAMPLE B

A cone has a base radius 3 in. and a volume of 24π in³. Find the height.



► **Solution**

Start with the volume formula and solve for H .

$$V = \frac{1}{3}BH$$

Volume formula for pyramids and cones.

$$V = \frac{1}{3}(\pi r^2) H$$

The base of a cone is a circle.

$$24\pi = \frac{1}{3}(\pi \cdot 3^2) H$$

Substitute 24π for the volume and 3 for the radius.

$$24\pi = 3\pi H$$

Square the 3 and multiply by $\frac{1}{3}$

$$8 = H$$

Solve for H .

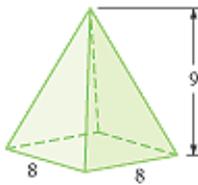
The height of the cone is 8 in.



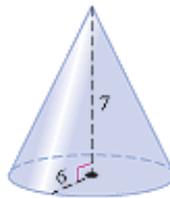
EXERCISES

Find the volume of each solid named in Exercises 1–6. All measurements are in centimeters.

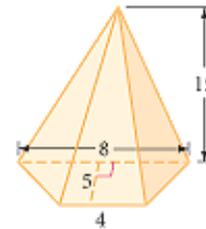
1. Square pyramid



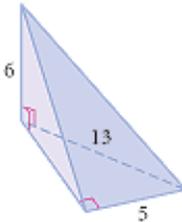
2. Cone



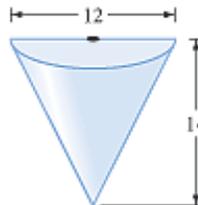
3. Trapezoidal pyramid



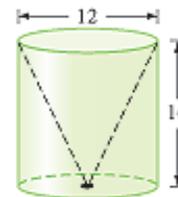
4. Triangular pyramid



5. Semicircular cone

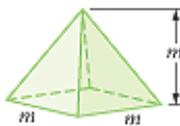


6. Cylinder with cone removed



In Exercises 7–9, express the total volume of each solid with the help algebra. In Exercise 9, what percentage of the volume is filled with the liquid? All measurements are in centimeters.

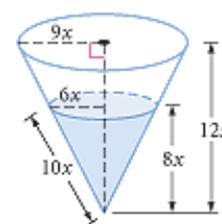
7. Square pyramid



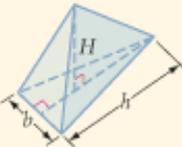
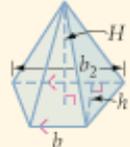
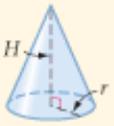
8. Cone



9. Cone



10. Use the information about the base and height each solid to find the volume.
All measurements are given in centimeters.

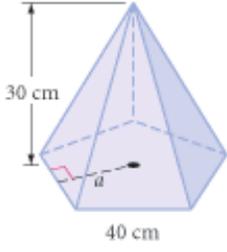
Information about base of solid	Height of solid	Triangular pyramid	Rectangular pyramid	Trapezoidal pyramid	Cone
$b = 6, b_2 = 7,$ $h = 6, r = 3$	$H = 20$				
$b = 9, b_2 = 22,$ $h = 8, r = 6$	$H = 20$	a. $V =$	d. $V =$	g. $V =$	j. $V =$
$b = 13, b_2 = 29,$ $h = 17, r = 8$	$H = 24$	b. $V =$	e. $V =$	h. $V =$	k. $V =$
		c. $V =$	f. $V =$	i. $V =$	l. $V =$

11. Sketch and label a square pyramid with height H feet and each side of the base M feet. The altitude meets the square base at the intersection of the two diagonals. Find the volume in terms H and M .
12. Sketch and label two different circular cones, each with a volume $2304 \pi \text{ cm}^3$.
13. Mount Fuji, the active volcano in Honshu, Japan, is 3776 m high and has a slope of approximately 30° . Mount Etna, in Sicily, is 3350 m high and approximately 50 km across the base. If you assume they both can be approximated by cones, which volcano is larger?

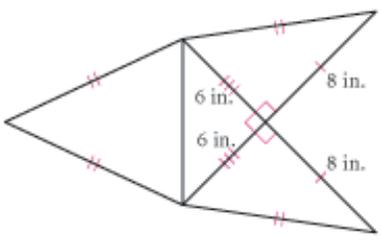


Mount Fuji is Japan's highest mountain. Legend claims that an earthquake created it.

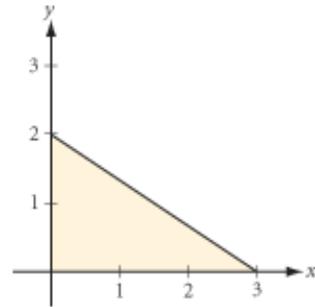
14. Bretislav has designed a crystal glass sculpture. Part of the piece is in the shape of a large regular pentagonal pyramid, shown at right. The apothem of the base measures 27.5 cm. How much will this part weigh if the glass he plans to use weighs 2.85 grams per cubic centimeter?



15. Jamala has designed a container that she claims will hold 50 in^3 . The net is shown at right. Check her calculations. What is the volume of the solid formed by this net? $\frac{1}{3}$

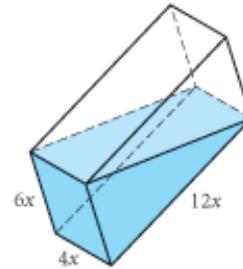


16. Find the volume of the solid formed by rotating the shaded figure about the x -axis.

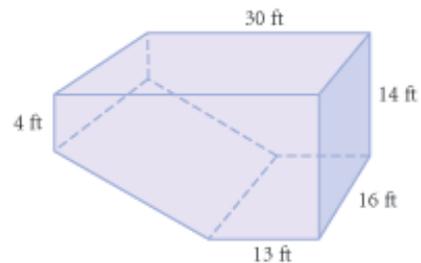


Review

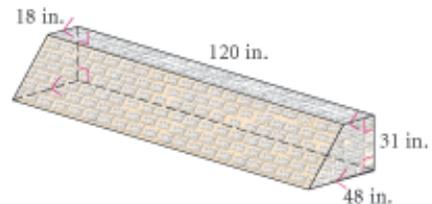
17. Find the volume of the liquid in this right rectangular prism. All measurements are given in centimeters.



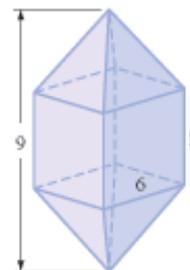
18. **Application** A swimming pool is in the shape of this prism. A cubic foot of water is about 7.5 gallons. How many gallons of water can the pool hold? If a pump is able to pump water into the pool at a rate of 15 gallons per minute, how long will it take to fill the pool?



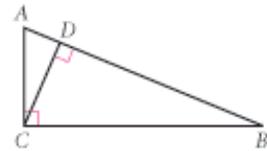
19. **Application** A landscape architect is building a stone retaining wall, as sketched at right. How many cubic feet of stone will she need?



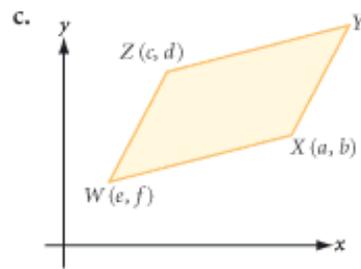
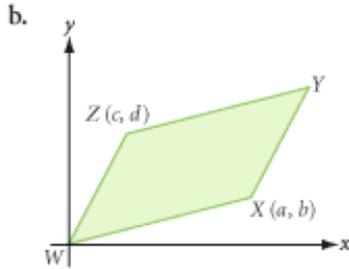
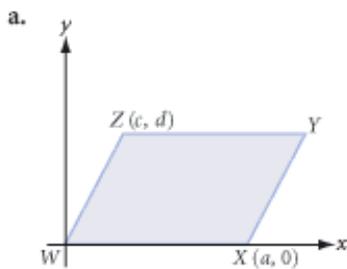
20. As bad as tanker oil spills are, they are only about 12% of the 3.5 million tons of oil that enters the oceans each year. The rest comes from routine tanker operations, sewage treatment plants' runoff, natural sources, and offshore oil rigs. One month's maintenance and routine operation of a single supertanker produces up to 17,000 gallons of oil sludge that gets into the ocean! If a cylindrical barrel is about 1.6 feet in diameter and 2.8 feet tall, how many barrels are needed to hold 17,000 gallons of oil sludge? Recall that a cubic foot of water is about 7.5 gallons.
21. Find the surface area of each of the following polyhedrons. (See the shapes on page 544.) Give *exact* answers.
- A regular tetrahedron with an edge 4 cm
 - A regular hexahedron with an edge 4 cm
 - A regular icosahedron with an edge 4 cm
 - The dodecahedron shown at right, made of four congruent rectangles and eight congruent triangles



22. Given the triangle at right, reflect D over \overline{AC} to D' . Then reflect D over \overline{BC} to D'' . Explain why D', C, D'' are collinear.



23. In each diagram, $WXYZ$ is a parallelogram. Find the coordinates of Y .



project

THE WORLD'S LARGEST PYRAMID

The pyramid at Cholula, Mexico, shown at right, was built between the 2nd and 8th centuries C.E. Like most pyramids of the Americas, it has a flat top. In fact, it is really two flat-topped pyramids.

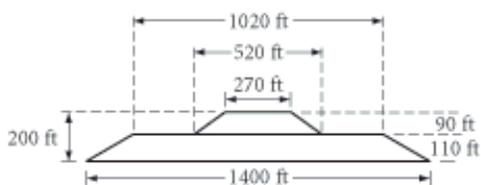
Some people claim it is the world's largest pyramid—even larger than the Great Pyramid of Khufu at Giza (shown on page 535) erected around 2500 B.C.E. Is it? Which of the two has the greater volume?

Your project should include

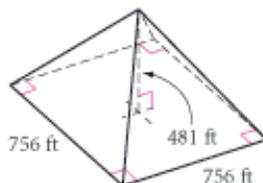
- ▶ Volume calculations for both pyramids.
- ▶ Scale models of both pyramids.



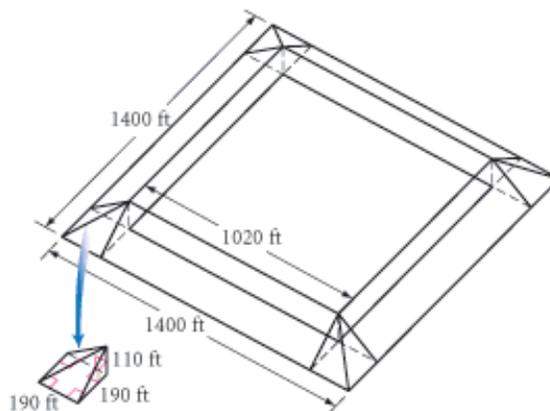
This church in Cholula appears to be on a hill, but it is actually built on top of an ancient pyramid!



Side view of the two pyramids at Cholula



Dimensions of the Great Pyramid at Giza



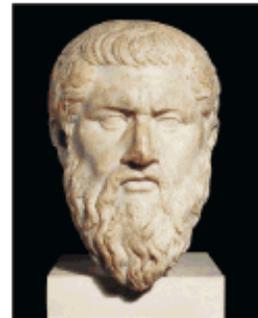
Dissected view of bottom pyramid at Cholula

Exploration

The Five Platonic Solids

Regular polyhedrons have intrigued mathematicians for thousands years. Greek philosophers saw the principles of mathematics and science as the guiding forces of the universe. Plato (429–347 B.C.E.) reasoned that because all objects are three-dimensional, their smallest parts must be in the shape regular polyhedrons. There are only five regular polyhedrons, and they are commonly called the **Platonic solids**.

Plato assigned each regular solid to one of the five “atoms”: the tetrahedron to fire, the icosahedron to water, the octahedron to air, the cube or hexahedron to earth, and the dodecahedron to the cosmos.



Plato

Fire



Regular tetrahedron
(4 faces)

Water



Regular icosahedron
(20 faces)

Air



Regular octahedron
(8 faces)

Earth



Regular hexahedron
(6 faces)

Cosmos



Regular dodecahedron
(12 faces)

Activity

Modeling the Platonic Solids

You will need

- poster board or cardboard
- a compass and straightedge
- a protractor
- scissors
- glue, paste, or cellophane tape
- colored pens or pencils for decorating the solids (*optional*)

What would each of the five Platonic solids look like when unfolded? There is more than one way to unfold each polyhedron. Recall that a flat figure that you can fold into a polyhedron is called its net.



Step 1 One face is missing in the net at right. Complete the net to show what the regular tetrahedron would look like if it were cut open along the three lateral edges and unfolded into one piece.



Step 2 Two faces are missing in the net at right. Complete the net to show what the regular hexahedron would look like if it were cut open along the lateral edges and three top edges, then unfolded.



Step 3 Here is one possible net for the regular icosahedron. When the net is folded together, the five top triangles meet at one top point. Which edge— a , b , or c —does the edge labeled x line up with?



Step 4 The regular octahedron is similar to the icosahedron but has only eight equilateral triangles. Complete the octahedron net at right. Two faces are missing.



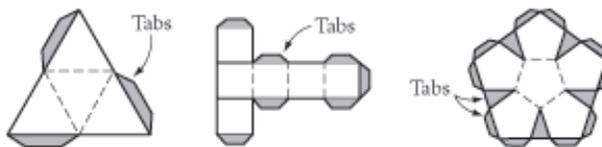
Step 5 The regular dodecahedron has 12 regular pentagons as faces. If you cut the dodecahedron into two equal parts, they would look like two flowers, each having five pentagon-shaped petals around a center pentagon. Complete the net for half a dodecahedron.



Now you know what the nets of the five Platonic solids could look like. Let's use the nets to construct and assemble models of the five Platonic solids. See the Procedure Note for some tips.

Procedure Note

1. To save time, build solids with the same shape faces at the same time.
2. Construct a regular polygon template for each shape face.
3. Erase the unnecessary line segments.
4. Leave tabs on some edges for gluing.



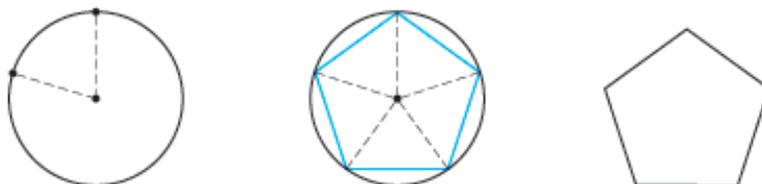
5. Decorate each solid before you cut it out.
6. Score on both sides of the net by running a pen or compass point over the fold lines.

Step 6 Build nets for the icosahedron, octahedron, and tetrahedron with equilateral triangles.

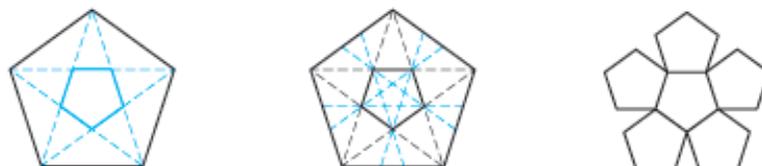
Step 7 Build a net for the hexahedron, or cube, with squares.

Step 8 Build a net for the dodecahedron with regular pentagons. Create half of the net, consisting of five pentagons surrounding a central pentagon, by following these steps.

- Construct a circle and place a point on the top the circle. Use your protractor to lightly draw a sector equal to one-fifth the circle. What must the measure of this central angle equal?
- Continue using your protractor to draw five congruent sectors. Check your work by measuring the last central angle formed. Connect the five points on the circle.
- Erase all radii and the circle. You now have one large regular pentagon.



- Lightly draw all of the diagonals. The smaller regular pentagon formed by the intersecting diagonals will be one of the 12 faces the dodecahedron.
- Lightly draw the diagonals of the central pentagon and extend them to the sides of the larger pentagon.
- Find the five pentagons that encircle the central pentagon and erase all other marks. You now have half of the net.



- Make a copy of the half net and assemble the two halves into the dodecahedron.

Step 9 Explore these extensions.

- Before you assemble your nets, decorate them according to the element that Plato assigned to each regular polyhedron, as described at the beginning of this exploration. Research how and why Plato assigned each of the five regular solids the elements Earth, Air, Fire, Water, and Cosmos.
- Can you explain from looking at the nets why there are only five Platonic solids? Search the Internet or library for a prothat there are only five regular polyhedrons.
- Complete a table of the vertices, faces, and edges the Platonic solids. Verify that they satisfy Euler's Formula for Polyhedrons. If a solid has a polyhedral hole, does the formula still hold?
- Use the Internet or library to research Platonic solids and dual solids. Does every Platonic solid have a dual? Is every dual of a Platonic solid also a Platonic solid?

Volume Problems

*If you have made mistakes...
there is always another
chance for you... for this
thing we call failure"
is not the falling down,
but the staying down.*

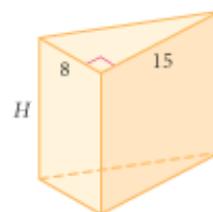
MARY PICKFORD

Volume has applications in science, medicine, engineering, and construction. For example, a chemist needs to accurately measure the volume of reactive substances. A doctor may need to calculate the volume of a cancerous tumor based on a body scan. Engineers and construction personnel need to determine the volume of building supplies such as concrete or asphalt. The volume of the rooms in a completed building will ultimately determine the size of mechanical devices such as air conditioning units.

Sometimes, if you know the volume of a solid, you can calculate an unknown length of a base or the solid's height. Here are two examples.

EXAMPLE A

The volume of this right triangular prism is 1440 cm^3 . Find the height of the prism.



► Solution

$$V = BH$$

$$V = \left(\frac{1}{2}bh\right)H$$

$$1440 = \frac{1}{2}(8)(15)H$$

$$1440 = 60H$$

$$24 = H$$

The height of the prism is 24 cm.

Volume formula for prisms and cylinders.

The base of the prism is a triangle.

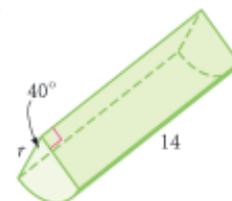
Substitute 1440 for the volume, 8 for the base of the triangle, and 15 for the height of the triangle.

Multiply.

Solve for H .

EXAMPLE B

The volume of this sector a right cylinder is 2814 m^3 . Find the radius of the base of the cylinder to the nearest m.



► Solution

The volume is the area of the base times the height. To find the area of the sector, you first find what fraction the sector is of the whole circle: $\frac{40}{360} = \frac{1}{9}$.

$$V = BH$$

$$V = \left(\frac{1}{9}\pi r^2\right)H$$

$$2814 = \frac{1}{9}\pi r^2 (14)$$

$$\frac{9 \cdot 2814}{14\pi} = r^2$$

$$575.8 \approx r^2$$

$$24 \approx r$$

The radius is about 24 m.

EXERCISES

You will need



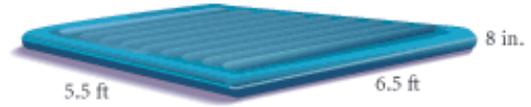
Construction tools
for Exercises 19 and 20

- If you cut a 1-inch square out of each corner of an 8.5-by-11-inch piece of paper and fold it into a box without a lid, what is the volume of the container?
- The prism at right has equilateral triangle bases with side lengths of 4 cm. The height of the prism is 8 cm. Find the volume.
- A triangular pyramid has a volume of 180 cm^3 and a height of 12 cm. Find the length of a side of the triangular base if the triangle's height from that side is 6 cm.
- A trapezoidal pyramid has a volume of 3168 cm^3 , and its height is 36 cm. The lengths of the two bases of the trapezoidal base are 20 cm and 28 cm. What is the height of the trapezoidal base?
- The volume of a cylinder is 628 cm^3 . Find the radius of the base if the cylinder has a height of 8 cm. Round your answer to the nearest 0.1 cm.
- If you roll an 8.5-by-11-inch piece of paper into a cylinder by bringing the two longer sides together, you get a tall, thin cylinder. If you roll an 8.5-by-11-inch piece of paper into a cylinder by bringing the two shorter sides together, you get a short, fat cylinder. Which of the two cylinders has the greater volume?
- Sylvia has just discovered that the valve on her cement truck failed during the night and that all the contents ran out to form a giant cone of hardened cement. To make an insurance claim, she needs to figure out how much cement is in the cone. The circumference of its base is 44 feet, and it is 5 feet high. Calculate the volume to the nearest cubic foot.
- A sealed rectangular container 6 cm by 12 cm by 15 cm is sitting on its smallest face. It is filled with water up to 5 cm from the top. How many centimeters from the bottom will the water level reach if the container is placed on its largest face?
- To test his assistant, noted adventurer Dakota Davis states that the volume of the regular hexagonal ring at right is equal to the volume of the regular hexagonal hole in its center. The assistant must confirm or refute this, using dimensions shown in the figure. What should he say to Dakota?

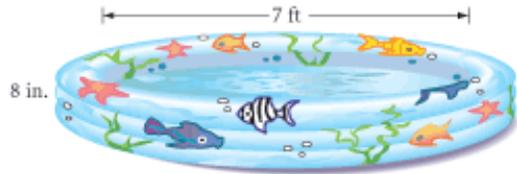


Use this information to solve Exercises 10–12: Water weighs about 63 pounds per cubic foot, and a cubic foot of water is about 7.5 gallons.

10. **Application** A king-size waterbed mattress measures 5.5 feet by 6.5 feet by 8 inches deep. To the nearest pound, how much does the water in this waterbed weigh? **h**



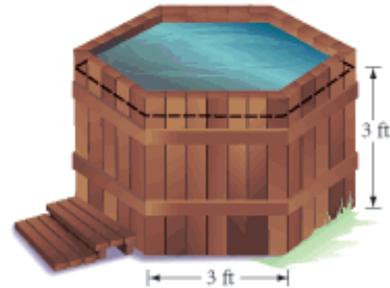
11. A child's wading pool has a diameter of 7 feet and is 8 inches deep. How many gallons of water can the pool hold? Round your answer to the nearest 0.1 gallon.



12. Madeleine's hot tub has the shape of a regular hexagonal prism. The chart on the hot-tub heater tells how long it takes to warm different amounts of water by 10°F. Help Madeleine determine how long it will take to raise the water temperature from 93°F to 103°F.

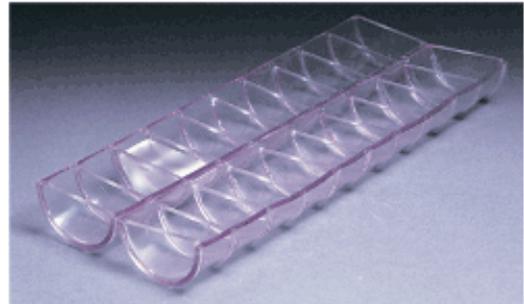
Minutes to Raise Temperature 10°F

Gallons	350	400	450	500	550	600	650	700
Minutes	9	10	11	12	14	15	16	18



13. A standard juice box holds 8 fluid ounces. A fluid ounce of liquid occupies 1.8 in^3 . Design a cylindrical can that will hold about the same volume as one juice box. What are some possible dimensions of the can?

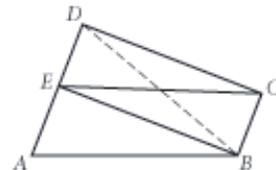
14. The photo at right shows an ice tray that is designed for a person who has the use of only one hand—each piece of ice will rotate out of the tray when pushed with one finger. Suppose the tray has a length of 12 inches and a height of 1 inch. Approximate the volume of water the tray holds if it is filled to the top. (Ignore the thickness of the plastic.)



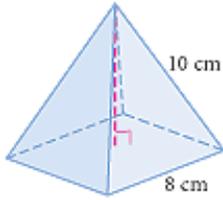
15. **Application** An auto tunnel through a mountain is being planned. It will be in the shape of a semicircular cylinder with a radius of 10 m and a length of 2 km. How many cubic meters of dirt will need to be removed? If the bed of each dump truck has dimensions 2.2 m by 4.8 m by 1.8 m, how many loads will be required to carry away the dirt?

Review

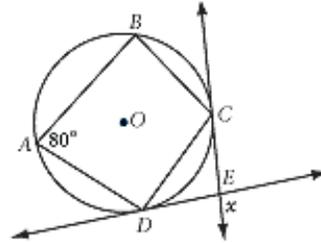
16. **Developing Proof** In the figure at right, $ABCE$ is a parallelogram and $BCDE$ is a rectangle. Write a paragraph proof showing that $\triangle ABD$ is isosceles.



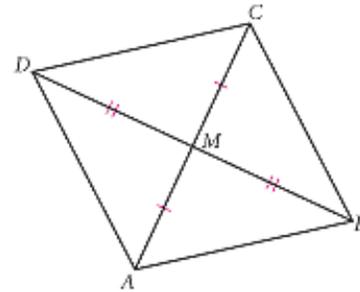
17. Find the height of this right square pyramid. Give your answer to the nearest 0.1 cm.



18. \overline{EC} is tangent at C . \overline{ED} is tangent at D . $m\angle C = 76^\circ$. Find x .



19. **Construction** Use your geometry tools to construct an inscribed and circumscribed circle for an equilateral triangle.
20. **Construction** Use your compass and straightedge to construct an isosceles trapezoid with a base angle of 45° and the length of one base three times the length of the other base.
21. M is the midpoint of \overline{AC} and \overline{BD} . For each statement, select always (A), sometimes (S), or never (N).
- $ABCD$ is a parallelogram.
 - $ABCD$ is a rhombus.
 - $ABCD$ is a kite.
 - $\triangle AMD \cong \triangle AMB$
 - $\angle DAM \cong \angle BCM$



IMPROVING YOUR REASONING SKILLS

Bert's Magic Hexagram

Bert is the queen's favorite jester. He entertains himself with puzzles. Bert is creating a magic hexagram on the front of a grid of 19 hexagons. When Bert's magic hexagram (like its cousin the magic square) is completed, it will have the same sum in every straight hexagonal row, column, or diagonal (whether it is three, four, or five hexagons long). For example, $B + 12 + 10$ is the same sum as $B + 2 + 5 + 6 + 9$, which is the same sum as $C + 8 + 6 + 11$. Bert planned to use just the first 19 positive integers (his age in years), but he only had time to place the first 12 integers before he was interrupted. Your job is to complete Bert's magic hexagram. What are the values for $A, B, C, D, E, F,$ and G ?



Displacement and Density

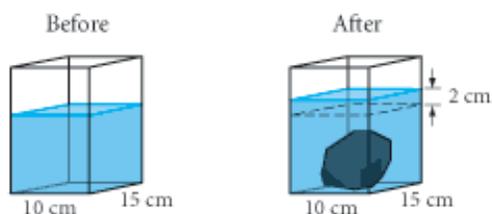
Eureka! I have found it!
ARCHIMEDES

What happens if you step into a bathtub that is filled to the brim? If you add a scoop of ice cream to a glass filled with root beer? In each case, you'll have a mess! The volume of the liquid that overflows in each case equals the volume of the solid below the liquid level. This volume is called an object's **displacement**.



EXAMPLE A

Mary Jo wants to find the volume of an irregularly shaped rock. She puts some water into a rectangular prism with a base that measures 10 cm by 15 cm. When the rock is put into the container, Mary Jo notices that the water level rises 2 cm because the rock displaces its volume of water. This new “slice” of water has a volume of $(2)(10)(15)$, or 300 cm^3 . So the volume of the rock is 300 cm^3 .



An important property of a material is its density. **Density** is the mass of matter in a given volume. You can find the mass of an object by weighing it. You calculate density by dividing the mass by the volume:

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

EXAMPLE B

A clump of metal with mass 351.4 grams is dropped into a cylindrical container, causing the water level to rise 1.1 cm. The radius of the base of the container is 3.0 cm. What is the density of the metal? Given the table, and assuming the metal is pure, what is the metal?

Metal	Density	Metal	Density
Aluminum	2.81 g/cm ³	Nickel	8.89 g/cm ³
Copper	8.97 g/cm ³	Platinum	21.40 g/cm ³
Gold	19.30 g/cm ³	Potassium	0.86 g/cm ³
Lead	11.30 g/cm ³	Silver	10.50 g/cm ³
Lithium	0.54 g/cm ³	Sodium	0.97 g/cm ³

► **Solution**

First, find the volume of displaced water. Then divide the mass by the volume to get the density of the metal.

$$\begin{aligned} \text{volume} &= \pi(3.0)^2 (1.1) & \text{density} &\approx \frac{351.4}{31.1} \\ &= (\pi) (9)(1.1) & &\approx 11.3 \\ &\approx 31.1 \end{aligned}$$

The density is 11.3 g/cm^3 . Therefore the metal is lead.

History

CONNECTION

Archimedes solved the problem of how to tell whether a crown was made of genuine gold by using displacement to calculate its density. Legend has it that the insight came to him while he was bathing. Thrilled by his discovery, Archimedes ran through the streets shouting “Eureka!” (Greek for “I have found it!”) wearing just what he’d been wearing in the bathtub.



EXERCISES

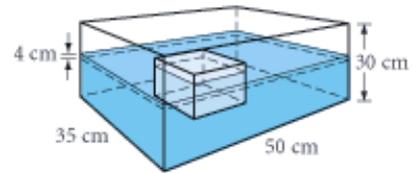
1. When you put a rock into a container of water, it raises the water level 3 cm. If the container is a rectangular prism whose base measures 15 cm by 15 cm, what is the volume of the rock?
2. You drop a solid glass ball into a cylinder with a radius of 6 cm, raising the water level 1 cm. What is the volume of the glass ball?
3. A fish tank 10 by 14 by 12 inches high is the home of a large goldfish named Columbia. She is taken out when her owner cleans the tank, and the water level in the tank drops $\frac{1}{3}$ inch. What is Columbia’s volume?



For Exercises 4–9, refer to the table on page 551.

4. What is the mass of a solid block of aluminum if its dimensions are 4 cm by 8 cm by 20 cm?
5. Which has more mass: a solid cylinder of gold with a height of 5 cm and a diameter of 6 cm or a solid cone of platinum with a height of 21 cm and a diameter of 8 cm?
6. Chemist Dean Dalton is given a clump of metal and is told that it is sodium. He finds that the metal has mass 145.5 g. He places it into a nonreactive liquid in a square prism whose base measures 10 cm on each edge. If the metal is indeed sodium, how high should the liquid level rise? h
7. A square-prism container with a base 5 cm by 5 cm is partially filled with water. You drop a clump of metal with mass 525 g into the container, and the water level rises 2 cm. What is the density of the metal? Assuming the metal is pure, what is the metal?

8. When ice floats in water, one-eighth of its volume floats above the water level and seven-eighths floats beneath the water level. A block of ice placed into an ice chest causes the water in the chest to rise 4 cm. The right rectangular chest measures 35 cm by 50 cm by 30 cm high. What is the volume of the block of ice? 



Science CONNECTION

Buoyancy is the tendency of an object to float in either a liquid or a gas. For an object to float on the surface of water, it must sink enough to displace the volume of water equal to its weight.



9. Sherlock Holmes rushes home to his chemistry lab, takes a mysterious medallion from his case, and weighs it. “It has mass 3088 grams. Now let’s check its volume.” He pours water into a graduated glass container with a 10-by-10 cm square base, and records the water level, which is 53.0 cm. He places the medallion into the container and reads the new water level, 54.6 cm. He enjoys a few minutes of mental calculation, then turns to Dr. Watson. “This confirms my theory. Quick, Watson! Off to the train station.”

“Holmes, you amaze me. Is it gold?” questions the good doctor.

“If it has a density of 19.3 grams per cubic centimeter, it is gold,” smiles Mr. Holmes. “If it is gold, then Colonel Banderson is who he says he is. If it is a fake, then so is the Colonel.”

“Well?” Watson queries.

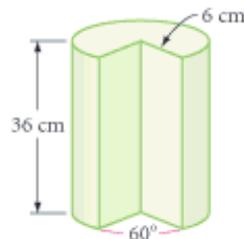
Holmes smiles and says, “It’s elementary, my dear Watson. Elementary geometry, that is.”

What is the volume of the medallion? Is it gold? Is Colonel Banderson who he says he is?

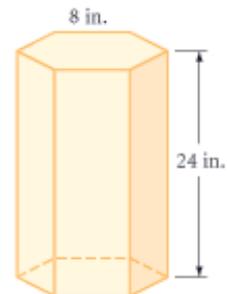


Review

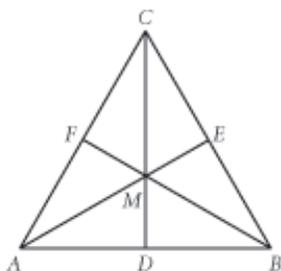
10. What is the volume of the slice removed from this right cylinder? Give your answer to the nearest cm^3 .



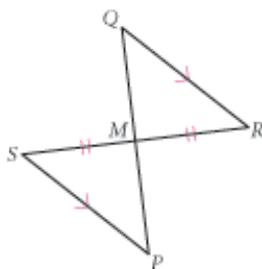
11. **Application** Ofelia has brought home a new aquarium shaped like the regular hexagonal prism shown at right. She isn’t sure her desk is strong enough to hold it. The aquarium, without water, weighs 48 pounds. How much will it weigh when it is filled? (Water weighs 63 pounds per cubic foot.) If a small fish needs about 180 cubic inches of water to swim around in, about how many small fish can this aquarium house?



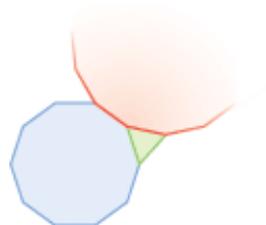
12. $\triangle ABC$ is equilateral .
 M is the centroid .
 $AB = 6$
 Find the area of $\triangle CEA$.



13. **Developing Proof** Give a paragraph or flowchart proof explaining why M is the midpoint of \overline{PQ} .



14. The three polygons are regular polygons. How many sides does the red polygon have?



15. A circle passes through the three points $(4, 7)$, $(6, 3)$, and $(1, -2)$.
 a. Find the center. b. Find the equation.
16. A secret rule matches the following numbers:
 $2 \rightarrow 4, 3 \rightarrow 7, 4 \rightarrow 10, 5 \rightarrow 13$
 Find $20 \rightarrow ?$, and $n \rightarrow ?$.

project

MAXIMIZING VOLUME

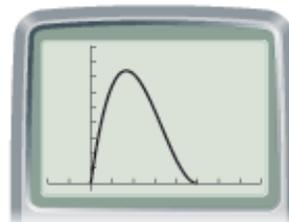
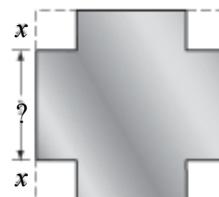
Suppose you have a 10-inch-square sheet of metal and you want to make a small box by cutting out squares from the corners of the sheet and folding up the sides. What size corners should you cut out to get the biggest box possible?

To answer this question, consider the length of the corner cut x and write an equation for the volume of the box, y , in terms of x . Graph your equation using reasonable window values. You should see your graph touch the x -axis in at least two places and reach a maximum somewhere in between. Study these points carefully to find their significance.

Your project should include

- ▶ The equation you used to calculate volume in terms of x and y .
- ▶ A sketch of the calculator graph and the graphing window you used.
- ▶ An explanation of important points, for example, when the graph touches the x -axis.
- ▶ A solution for what size corners make the biggest volume.

Lastly, generalize your findings. For example, what fraction of the side length could you cut from each corner of a 12-inch-square sheet to make a box of maximum volume?



Exploration

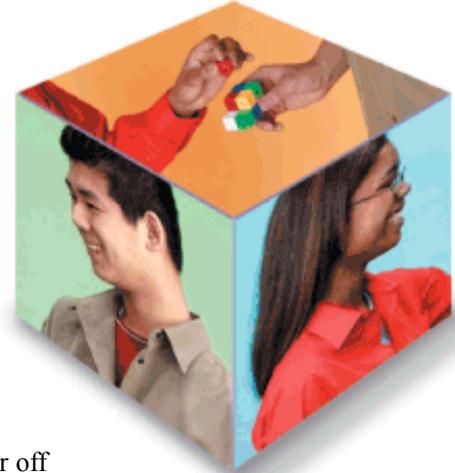
Orthographic Drawing

If you have ever put together a toy from detailed instructions, or built a birdhouse from a kit, or seen blueprints for a building under construction, you have seen isometric drawings.

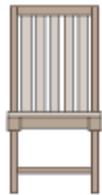
Isometric means “having equal measure,” so the edges of a cube drawn isometrically all have the same length. In contrast, recall that when you drew a cube in two-point perspective, you needed to use edges of different lengths to get a natural look.

When you buy a product from a catalog or off the Internet, you want to see it from several angles. The top, front, and right side views are given in an

orthographic drawing. Ortho means “straight,” and the views of an orthographic drawing show the faces of a solid as though you are looking at them “head-on.”



Top



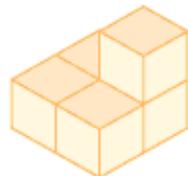
Front



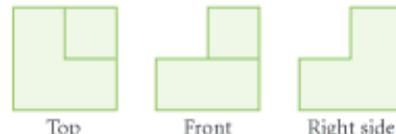
Side



Isometric



An isometric drawing



An orthographic drawing

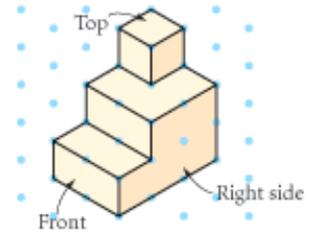
Career CONNECTION

Architects create blueprints for their designs, as shown at right. Architectural drawing plans use orthographic techniques to describe the proposed design from several angles. These front and side views are called building elevations.



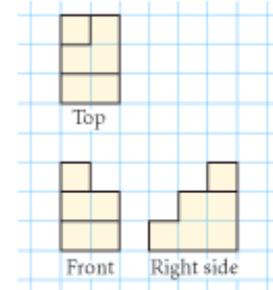
EXAMPLE A

Make an orthographic drawing of the solid shown in the isometric drawing at right.



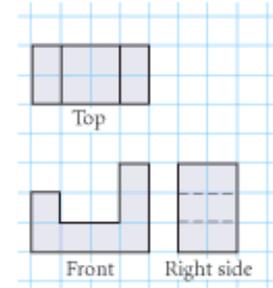
► **Solution**

Visualize how the solid would look from the top, the front, and the right side. Draw an edge wherever there is a change of depth. The top and front views must have the same width, and the front and right side views must have the same height.



EXAMPLE B

Draw the isometric view of the object shown here as an orthographic drawing. The dashed lines mean that there is an invisible edge.



► **Solution**

Find the vertices of the front face and make the shape. Use the width of the side and top views to extend parallel lines. Complete the back edges. You can shade parallel planes to show depth.



British pop artist David Hockney (b 1937) titled this photographic collage *Sunday Morning Mayflower Hotel, N.Y., Nov-28, 1982*. Each photo shows the view from a different angle, just as an orthographic drawing shows multiple angles at once.



Activity

Isometric and Orthographic Drawings

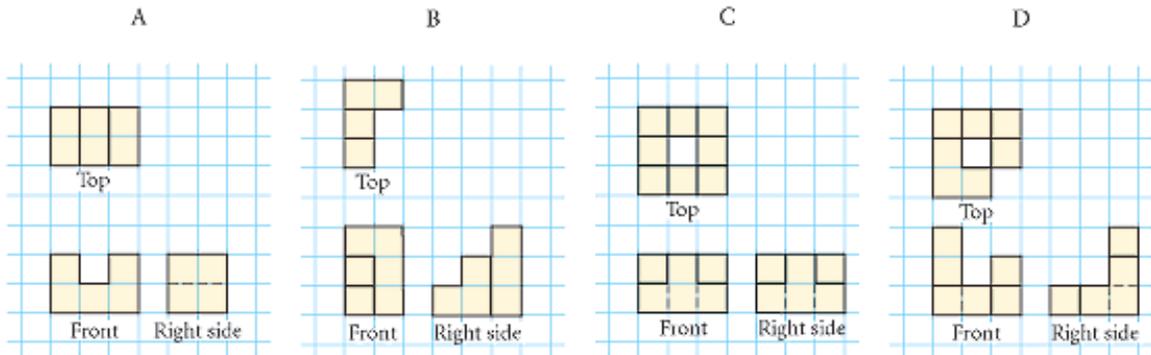
You will need

- isometric dot paper
- graph paper
- 12 cubes

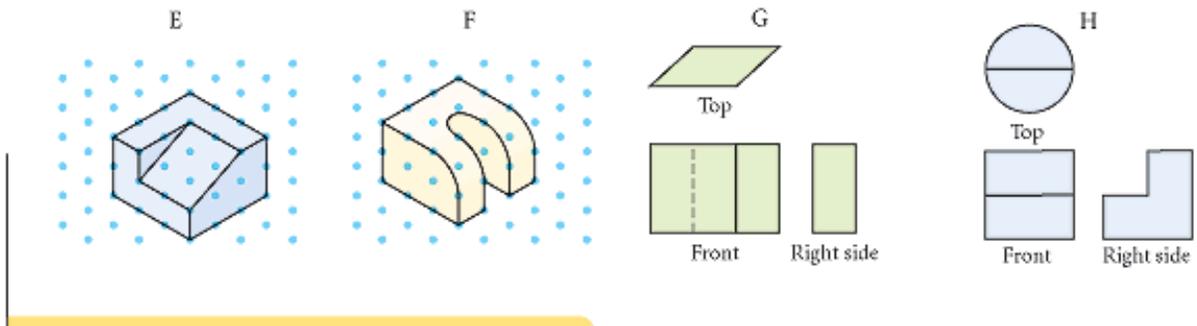
In this investigation you'll build block models and draw their isometric and orthographic views.



- Step 1** Practice drawing a cube on isometric dot paper. What is the shape of each visible face? Are they congruent? What should the orthographic views of a cube look like?
- Step 2** Stack three cubes to make a two-step "staircase." Turn the structure so that you look at it the way you would walk up stairs. Call that view the front. Next, identify the top and right sides. How many planes are visible from each view? Make an isometric drawing of the staircase on dot paper and the three orthographic views on graph paper.
- Step 3** Build solids A–D from their orthographic views, then draw their isometric views.



- Step 4** Make your own original 8- to 12-cube structure and agree on the orthographic views that represent it. Then trade places with another group and draw the orthographic views of their structure.
- Step 5** Make orthographic views for solids E and F, and sketch the isometric views of solids G and H.



Volume of a Sphere

Satisfaction lies in the effort, not in the attainment.

Full effort is full victory.

MOHANDAS K. GANDHI

In this lesson you will develop a formula for the volume of a sphere. In the investigation you'll compare the volume of a right cylinder to the volume of a hemisphere.



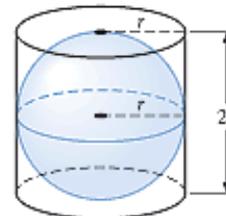
Investigation The Formula for the Volume of a Sphere

You will need

- cylinder and hemisphere with the same radius
- sand, rice, birdseed, or water

This investigation demonstrates the relationship between the volume of a hemisphere with radius r and the volume of a right cylinder with base radius r and height $2r$ —that is, the smallest cylinder that encloses a given sphere.

- Step 1 Fill the hemisphere.
- Step 2 Carefully pour the contents of the hemisphere into the cylinder. What fraction of the cylinder does the hemisphere appear to fill?
- Step 3 Fill the hemisphere again and pour the contents into the cylinder. What fraction of the cylinder do two hemispheres (one sphere) appear to fill?
- Step 4 If the radius of the cylinder is r and its height is $2r$, then what is the volume of the cylinder in terms of r ?
- Step 5 The volume of the sphere is the fraction of the cylinder's volume that was filled by two hemispheres. What is the formula for the volume of a sphere? State it as your conjecture.



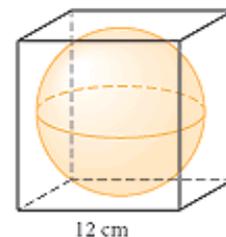
Sphere Volume Conjecture

C-88

The volume of a sphere with radius r is given by the formula $\underline{\quad ? \quad}$.

EXAMPLE A

As an exercise for her art class, Mona has cast a plaster cube 12 cm on each side. Her assignment is to carve the largest possible sphere from the cube. What percentage of the plaster will be carved away?

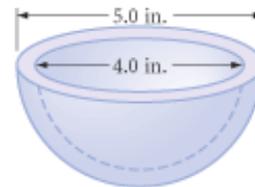


► **Solution**

The largest possible sphere will have a diameter of 12 cm, so its radius is 6 cm. Applying the formula for volume of a sphere, you get $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(6)^3 = \frac{4}{3}\pi \cdot 216 = 288\pi$, or about 905 cm^3 . The volume of the plaster cube is 12^3 , or 1728 cm^3 . You subtract the volume of the sphere from the volume of the cube to get the amount carved away, which is about 823 cm^3 . Therefore the percentage carved away is $\frac{823}{1728} \approx 48\%$.

EXAMPLE B

Find the volume of plastic (to the nearest cubic inch) needed for this hollow toy component. The outer-hemisphere diameter is 5.0 in. and the inner-hemisphere diameter is 4.0 in.



► **Solution**

The formula for volume of a sphere is $V = \frac{4}{3}\pi r^3$, so the volume of a hemisphere is half of that, $V = \frac{2}{3}\pi r^3$. A radius is half a diameter.

Outer Hemisphere

$$\begin{aligned} V_o &= \frac{2}{3}\pi r^3 \\ &= \frac{2}{3}\pi (2.5)^3 \\ &= \frac{2}{3}\pi \cdot 15.625 \\ &= \frac{31.25\pi}{3} \approx 32.7 \end{aligned}$$

Inner Hemisphere

$$\begin{aligned} V_i &= \frac{2}{3}\pi r^3 \\ &= \frac{2}{3}\pi (2)^3 \\ &= \frac{2}{3}\pi \cdot 8 \\ &= \frac{16\pi}{3} \approx 16.8 \end{aligned}$$

Subtracting the volume of the inner hemisphere from the volume of the outer one, approximately 16 in^3 of plastic are needed.



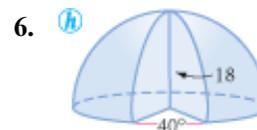
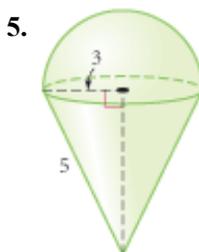
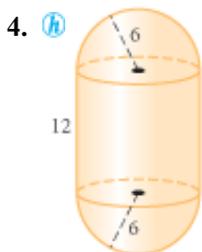
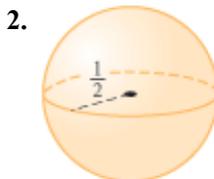
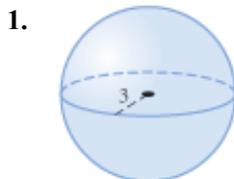
EXERCISES

You will need



Geometry software for Exercises 21 and 22

In Exercises 1–6, find the volume of each solid. All measurements are in centimeters.



7. What is the volume of the largest hemisphere that you could carve out of a wooden block whose edges measure 3 m by 7 m by 7 m?

8. A sphere of ice cream is placed onto your ice cream cone. Both have a diameter of 8 cm. The height of your cone is 12 cm. If you push the ice cream into the cone, will all of it fit?

9. **Application** Lickety Split ice cream comes in a cylindrical container with an inside diameter of 6 inches and a height of 10 inches. The company claims to give the customer 25 scoops of ice cream per container, each scoop being a sphere with a 3-inch diameter. How many scoops will each container really hold?



10. Find the volume of a spherical shell with an outer diameter of 8 meters and an inner diameter of 6 meters.

11. Which is greater, the volume of a hemisphere with radius 2 cm or the total volume of two cones with radius 2 cm and height 2 cm?

12. A sphere has a volume of 972π in³. Find its radius. 

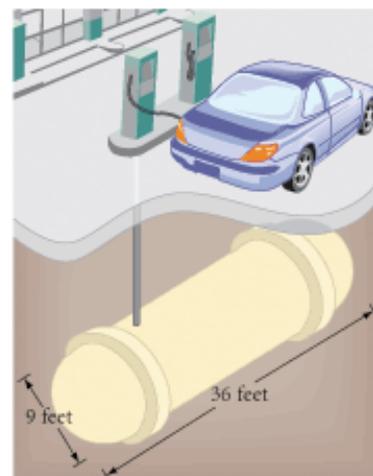
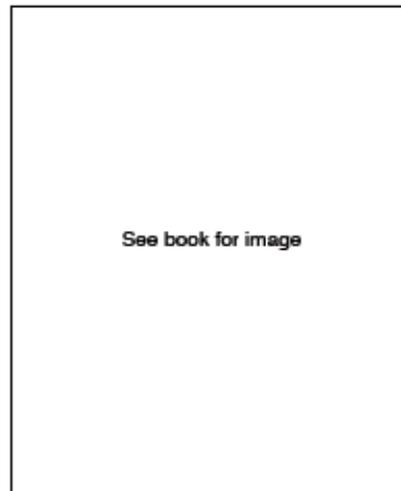
13. A hemisphere has a volume of 18π cm³. Find its radius.

14. The base of a hemisphere has an area of 256π cm². Find its volume.

15. If the diameter of a student's brain is about 6 inches, and you assume its shape is approximately a hemisphere, then what is the volume of the student's brain?

16. A cylindrical glass 10 cm tall and 8 cm in diameter is filled to 1 cm from the top with water. If a golf ball 4 cm in diameter is placed into the glass, will the water overflow?

17. **Application** This underground gasoline storage tank is a right cylinder with a hemisphere at each end. How many gallons of gasoline will the tank hold? (1 gallon = 0.13368 cubic foot.) If the service station fills twenty 15-gallon tanks from the storage tank per day, how many days will it take to empty the storage tank?



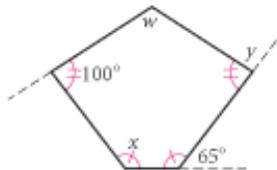
Review

18. Inspector Lestrade has sent a small piece of metal to the crime lab. The lab technician finds that its mass is 54.3 g. It appears to be lithium, sodium, or potassium, all highly reactive with water. Then the technician places the metal into a graduated glass cylinder of radius 4 cm that contains a nonreactive liquid. The metal causes the level of the liquid to rise 2.0 cm. Which metal is it? (Refer to the table on page 551.)
19. City law requires that any one-story commercial building supply a parking area equal in size to the floor area of the building. A-Round Architects has designed a cylindrical building with a 150-foot diameter. They plan to ring the building with parking. How far from the building should the parking lot extend? Round your answer to the nearest foot.
20. Plot A , B , C , and D onto graph paper.
 A is $(3, -5)$.
 C is the reflection of A over the x -axis.
 B is the rotation of C 180° around the origin.
 D is a transformation of A by the rule $(x, y) \rightarrow (x + 6, y + 10)$.



What kind of quadrilateral is $ABCD$? Give reasons for your answer.

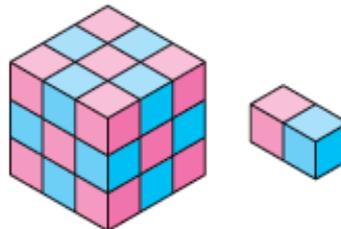
21. **Technology** Use geometry software to construct a segment \overline{AB} and its midpoint C . Trace \overline{AC} and \overline{BC} , and drag B around to sketch a shape. Compare the shapes they trace.
22. **Technology** Use geometry software to construct a circle. Choose a point A on the circle and a point B not on the circle, and construct the perpendicular bisector of \overline{AB} . Trace the perpendicular bisector as you animate A around the circle. Describe the locus of points traced.
23. Find w , x , and y .



IMPROVING YOUR VISUAL THINKING SKILLS

Patchwork Cubes

The large cube at right is built from 13 double cubes like the one shown plus one single cube. All faces are identical. What color must the single cube be, and where must it be positioned?



Surface Area of a Sphere

Sometimes it's better to talk about difficult subjects lying down; the change in posture sort of tilts the world so you can get a different angle on things.

MARY WILLIS WALKER

Earth is so large that it is reasonable to use area formulas for plane figures—rectangles, triangles, and circles—to find the areas of most small land regions. But, to find Earth's entire surface area, you need a formula for the surface area of a sphere. Now that you know how to find the volume of a sphere, you can use that knowledge to arrive at the formula for the surface area of a sphere.



Earth rises over the Moon's horizon. From there, the Moon's surface seems flat.



Investigation

The Formula for the Surface Area of a Sphere

In this investigation you'll visualize a sphere's surface covered by tiny shapes that are nearly flat. So the surface area, S , of the sphere is the sum of the areas of all the "near polygons." If you imagine radii connecting each of the vertices of the "near polygons" to the center of the sphere, you are mentally dividing the volume of the sphere into many "near pyramids." Each of the "near polygons" is a base for a pyramid, and the radius, r , of the sphere is the height of the pyramid. So the volume, V , of the sphere is the sum of the volumes of all the pyramids. Now get ready for some algebra.



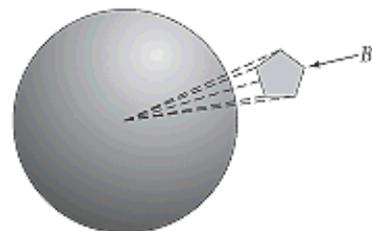
A horsefly's eyes resemble spheres covered by "near polygons."

- Step 1 Divide the surface of the sphere into 1000 "near polygons" with areas $B_1, B_2, B_3, \dots, B_{1000}$. Then you can write the surface area, S , of the sphere as the sum of the 1000 B 's:

$$S = B_1 + B_2 + B_3 + \dots + B_{1000}$$

- Step 2 The volume of the pyramid with base B_1 is $\frac{1}{3}(B_1)(r)$, so the total volume of the sphere, V , is the sum of the volumes of the 1000 pyramids:

$$V = \frac{1}{3}(B_1)(r) + \frac{1}{3}(B_2)(r) + \dots + \frac{1}{3}(B_{1000})(r)$$



- What common expression can you factor from each of the terms on the right side? Rewrite the last equation showing the results of your factoring.
- Step 3 But the volume of the sphere is $V = \frac{4}{3}\pi r^3$. Rewrite your equation from Step 2 by substituting $\frac{4}{3}\pi r^3$ for V and substituting for S the sum of the areas of all the “near polygons.”
- Step 4 Solve the equation from Step 3 for the surface area, S . You now have a formula for finding the surface area of a sphere in terms of its radius. State this as your next conjecture and add it to your conjecture list.

Sphere Surface Area Conjecture

C-89

The surface area, S , of a sphere with radius r is given by the formula $\frac{4}{3}\pi r^2$.

EXAMPLE Find the surface area of a sphere whose volume is $12,348\pi \text{ m}^3$.

► **Solution**

First, use the volume formula for a sphere to find its radius. Then use the radius to find the surface area.

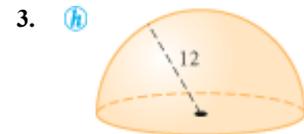
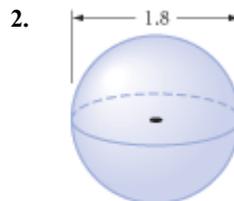
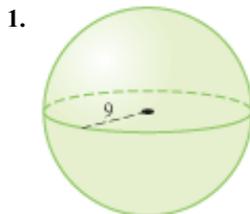
Radius Calculation	Surface Area Calculation
$V = \frac{4}{3}\pi r^3$	$S = 4\pi r^2$
$12,348\pi = \frac{4}{3}\pi r^3$	$= 4\pi(21)^2$
$\frac{3}{4} \cdot 12,348 = r^3$	$= 4\pi \cdot 441$
$9261 = r^3$	$S = 1764\pi \approx 5541.8$
$r = 21$	

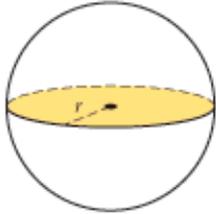
The radius is 21 m, and the surface area is $1764\pi \text{ m}^2$, or about 5541.8 m^2 .



EXERCISES

For Exercises 1–3, find the volume and total surface area of each solid. All measurements are in centimeters.



4. The shaded circle at right has area 40π cm². Find the surface area of the sphere. 

5. Find the volume of a sphere whose surface area is 64π cm².

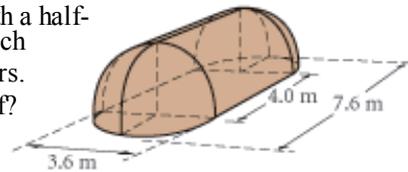
6. Find the surface area of a sphere whose volume is 288π cm³.

7. If the radius of the base of a hemisphere (which is bounded by a great circle) is r , what is the area of the great circle? What is the total surface area of the hemisphere, including the base? How do they compare?



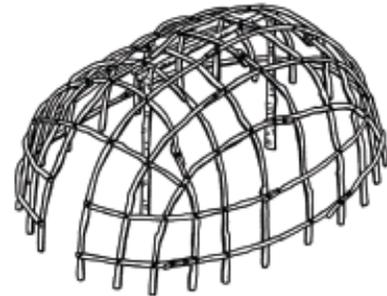
8. If Jose used 4 gallons of wood sealant to cover the hemispherical ceiling of his vacation home, how many gallons of wood sealant are needed to cover the floor?

9. **Application** Assume a Kickapoo wigwam is a semicylinder with a half-hemisphere on each end. The diameter of the semicylinder and each of the half-hemispheres is 3.6 meters. The total length is 7.6 meters. What is the volume of the wigwam and the surface area of its roof?



Cultural CONNECTION

A wigwam was a domed structure that Native American woodland tribes, such as the Kickapoo, Iroquois, and Cherokee, used for shelter and warmth in the winter. They designed each wigwam with an oval floor pattern, set tree saplings vertically into the ground around the oval, bent the tips of the saplings into an arch, and tied all the pieces together to support the framework. They then wove more branches horizontally around the building and added mats over the entire dwelling, except for the doorway and smoke hole.



10. **Application** A farmer must periodically resurface the interior (wall, floor, and ceiling) of his silo to protect it from the acid created by the silage. The height of the silo to the top of the hemispherical dome is 50 ft, and the diameter is 18 ft.

- What is the approximate surface area that needs to be treated?
- If 1 gallon of resurfacing compound covers about 250 ft², how many gallons are needed?
- There is 0.8 bushel per ft³. Calculate the number of bushels of grain this silo will hold.

11. About 70% of Earth's surface is covered by water. If the diameter of Earth is about 12,750 km, find the area not covered by water to the nearest 100,000 km².

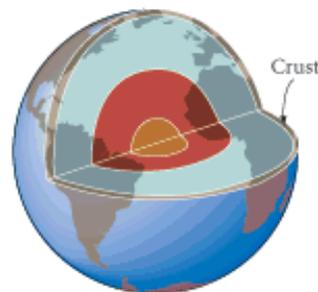


History CONNECTION

From the early 13th century to the late 17th century, the Medici family of Florence, Italy, were successful merchants and generous patrons of the arts. The Medici family crest, shown here, features six spheres—five red spheres and one that resembles the earth. The use of three gold spheres to advertise a pawnshop could have been inspired by the Medici crest.



12. A sculptor has designed a statue that features six hemispheres (inspired by the Medici crest) and three spheres (inspired by the pawnshop logo). He wants to use gold electroplating on the six hemispheres (diameter 6 cm) and the three spheres (diameter 8 cm), which will cost about $14\text{¢}/\text{cm}^2$. (The bases of the hemispheres will *not* be electroplated.) Will he be able to stay under his \$150 budget? If not, what diameter spheres should he make to stay under budget?
13. Earth has a thin outer layer called the *crust*, which averages about 24 km thick. Earth's diameter is about 12,750 km. What percentage of the volume of Earth is the crust?



Review

14. A piece of wood placed in a cylindrical container causes the container's water level to rise 3 cm. This type of wood floats half out of the water, and the radius of the container is 5 cm. What is the volume of the piece of wood?
15. Find the ratio of the area of the circle inscribed in an equilateral triangle to the area of the circumscribed circle.
16. Find the ratio of the area of the circle inscribed in a square to the area of the circumscribed circle.
17. Find the ratio of the area of the circle inscribed in a regular hexagon to the area of the circumscribed circle.
18. Make a conjecture as to what happens to the ratio in Exercises 15–17 as the number of sides of the regular polygon increases. Make sketches to support your conjecture.
19. Use inductive reasoning to complete each table.

a.

n	1	2	3	4	5	6	...	n	...	200
$f(n)$	-2	1	4	7			

b.

n	1	2	3	4	5	6	...	n	...	200
$f(n)$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$			

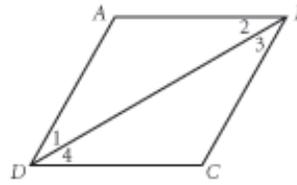
h

20. **Developing Proof** Prove the Rhombus Angles Conjecture: The diagonals of a rhombus bisect the angles of the rhombus.

Given: Rhombus $ABCD$ with diagonal \overline{BD}

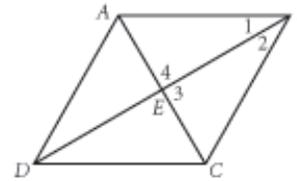
Show: \overline{BD} bisects $\angle ABC$ and $\angle ADC$

Explain why the logic of this proof would also apply to the other diagonal, \overline{AC} .



21. **Developing Proof** Prove the Rhombus Diagonals Conjecture: The diagonals of a rhombus are perpendicular bisectors of each other. Start with rhombus $ABCD$ in the figure below and follow these steps.

- Use the Rhombus Angles Conjecture to prove that $\triangle AEB \cong \triangle CEB$.
- Use part a to prove that \overline{BD} bisects \overline{AC} .
- Use part a to prove that $\angle 3$ and $\angle 4$ are right angles.
- You've proved most of the Rhombus Diagonals Conjecture. Explain what is missing and describe how you could complete the proof.



IMPROVING YOUR REASONING SKILLS

Reasonable 'rithmetic II

Each letter in these problems represents a different digit.

1. What is the value of C ?

$$\begin{array}{r} 8 \ 7 \ 8 \ 9 \\ 3 \ B \ A \ 7 \\ 4 \ 8 \ 2 \ A \\ + \ 7 \ A \ B \ 5 \\ \hline 2 \ C \ 2 \ 8 \ 7 \end{array}$$

2. What is the value of D ?

$$\begin{array}{r} D \ E \ F \ F \\ - \ E \ 2 \ F \ 6 \\ \hline 1 \ 9 \ 9 \ 7 \end{array}$$

3. What is the value of K ?

$$\begin{array}{r} GJ \\ 7 \overline{)HGK} \\ \underline{21} \\ HK \\ HK \end{array}$$

4. What is the value of N ?

$$\begin{array}{r} 5 \ 2 \\ LQ \overline{)NM2} \\ \underline{NP} \\ M2 \\ M2 \end{array}$$



Solving for Any Variable

Most formulas are defined to give you one variable in terms of other variables. But you can use algebra to rearrange the formula and define it in terms of any variable that you wish. Solving for a particular variable saves time when you have repeated calculations based on the same variables.

EXAMPLE

A chocolate manufacturer is experimenting with new cylindrical cans for hot-cocoa mix. The can needs to hold 73 in³ (roughly 40 oz). Find the can diameters that correspond to heights of 4 in., 5 in., 6 in., and 7 in.

► **Solution**

The formula for the volume of a cylinder is $V = \pi r^2 H$, or $V = \pi \left(\frac{d}{2}\right)^2 H$. Repeatedly substituting values for V and H into the formula and solving for d is tedious. Instead, solve for d once and then substitute values for the other variables.

$$V = \pi \left(\frac{d}{2}\right)^2 H \quad \text{The original formula.}$$

$$\frac{V}{\pi} = \left(\frac{d}{2}\right)^2 H \quad \text{Divide both sides by } \pi.$$

$$\frac{V}{\pi H} = \left(\frac{d}{2}\right)^2 \quad \text{Divide both sides by } H.$$

$$\sqrt{\frac{V}{\pi H}} = \frac{d}{2} \quad \text{Take the positive square root of both sides.}$$

$$2\sqrt{\frac{V}{\pi H}} = d \quad \text{Multiply both sides by 2.}$$

To recall a formula that you've typed into a graphing calculator, press **2nd** **ENTER** and change any values. This is helpful for finding values in a table.

Now use the formula to calculate the diameters. It may help to organize your information in a table. Remember that the volume in this situation is always 73 in³.

The corresponding can diameters are approximately 4.8 in., 4.3 in., 3.9 in., and 3.6 in.

H	$d = 2\sqrt{\frac{V}{\pi H}}$
4	$d = 2\sqrt{\frac{73}{\pi(4)}} \approx 4.8$
5	$d = 2\sqrt{\frac{73}{\pi(5)}} \approx 4.3$
6	$d = 2\sqrt{\frac{73}{\pi(6)}} \approx 3.9$
7	$d = 2\sqrt{\frac{73}{\pi(7)}} \approx 3.6$



EXERCISES

In Exercises 1–6, solve the formula for the given variable. These formulas should be familiar from geometry.

- 1. $A = bh$ for h
- 2. $P = 2b + 2h$ for b
- 3. $V = \frac{1}{3}\pi r^2 H$ for r
- 4. $a^2 + b^2 = c^2$ for b
- 5. $SA = \frac{1}{2}P(l + a)$ for a
- 6. $m = \frac{y_2 - y_1}{x_2 - x_1}$ for y_2

In Exercises 7–9, solve the formula for the given variable. These formulas may be familiar from science. Can you identify what each original formula is used for?

- 7. $d = vt$ for v
- 8. $C = \frac{5}{9}(F - 32)$ for F
- 9. $T = 2\sqrt{\frac{L}{g}}$ for L

10. In the Exploration Euler’s Formula for Polyhedrons, you discovered a formula that relates the number of vertices (V), edges (E), and faces (F) of a polyhedron. One version of Euler’s formula is $F = E - V + 2$.

- a. Solve Euler’s formula for V .
- b. Use the formula for V to find the number of vertices that each polyhedron listed in the table must have.

	Pentahedron	Hexahedron	Octahedron	Decahedron	Dodecahedron
Number of faces					
Number of edges	8	10	12	16	20
Number of vertices					

- 11. A manufacturer of ice cream cones wants to field-test cones of different sizes that each require the same amount of dough to make. The manufacturer’s standard cone uses a surface area of about 114 cm² of dough.
 - a. Recall that the lateral surface area of a right cone is $SA = \pi rl$, where r is the radius of the base and l is the slant height. Find the cone radii that correspond to slant heights of 10 cm, 11 cm, 12 cm, and 13 cm.
 - b. Notice that the radius, height, and slant height form a right triangle, so $r^2 + H^2 = l^2$. Use your results from part a to find the height of each cone.
 - c. Calculate the volume of each cone to the nearest cubic centimeter.
 - d. As a consumer, which cone do you think you would prefer? Explain.
 - e. If you were the owner of an ice cream parlor, which cone do you think you would prefer? Explain.
- 12. In Kara’s university biology class, her overall average is calculated as the average of five exams: three midterm exams and a final exam that counts twice. The maximum score on any exam is 100 points.
 - a. Let m_1 , m_2 , and m_3 represent scores on the three midterm exams, and let f represent the score on the final exam. Write a formula for Kara’s overall average, A .
 - b. Kara has already scored a 75, 80, and 83 on the three midterm exams. Find the score she must earn on the final exam to have an overall average of 60, 70, 80, or 90.



Exploration

Sherlock Holmes and Forms of Valid Reasoning



If the integer is divisible by 6, then it is divisible by 3. The integer is not divisible by 3. Therefore, it is not divisible by 6.

“That’s logical!” You’ve probably heard that expression many times. What do we mean when we say someone is thinking logically? One dictionary defines *logical* as “capable of reasoning or using reason in an orderly fashion that brings out fundamental points.”

“Prove it!” That’s another expression you’ve probably heard many times. It is an expression that is used by someone concerned with logical thinking. In daily life, proving something often means you can present some facts to support a point.

The fictional character Sherlock Holmes, created by Sir Arthur Conan Doyle, was known as the master of deductive reasoning. The reasoning about his sidekick Watson in the examples and exercises below is adapted from a Sherlock Holmes story *The Adventure of the Dancing Men*.

When you apply deductive reasoning, you are “being logical” like detective Sherlock Holmes. The statements you take as true are called premises, and the statements that follow from them are conclusions.

When you translate a deductive argument into symbolic form, you use capital letters to stand for simple statements. When you write “If P then Q ,” you are writing a **conditional statement**. Here are two examples.



$P \rightarrow Q$
 $\sim Q$
 $\therefore \sim P$

English argument	Symbolic translation
If Watson has chalk between his fingers, then he has been playing billiards. Watson has chalk between his fingers. Therefore Watson has been playing billiards.	P : Watson has chalk between his fingers. Q : Watson has been playing billiards. If P then Q . P $\therefore Q$
If $\triangle ABC$ is isosceles, then the base angles are congruent. Triangle ABC is isosceles. Therefore its base angles are congruent.	P : Triangle ABC is isosceles. Q : Triangle ABC 's base angles are congruent. If P then Q . P $\therefore Q$

The symbol \therefore means “therefore.” So you can read the last two lines “ $P, \therefore Q$ ” as “ P , therefore Q ” or “ P is true, so Q is true.”

Both of these examples illustrate one of the well-accepted forms of valid reasoning. According to **Modus Ponens** (MP), if you accept “if P then Q ” as true and you accept P as true, then you must logically accept Q as true.



In geometry—as in daily life—we often encounter “not” in a statement. “Not P ” is the **negation** of statement P . If P is the statement “It is raining,” then “not P ,” symbolized $\sim P$, is the statement “It is not raining” or “It is not the case that it is raining.” To remove negation from a statement, you remove the not. The negation of the statement “It is not raining” is “It is raining.” You can also negate a “not” by adding yet another “not.” So you can also negate the statement “It is not raining” by saying “It is not the case that it is not raining.” This property is called **double negation**.

According to **Modus Tollens** (MT), if you accept “if P then Q ” as true and you accept $\sim Q$ as true, then you must logically accept $\sim P$ as true. Here are two examples.

English argument	Symbolic translation
If Watson wished to invest money with Thurston, then he would have had his checkbook with him. Watson did not have his checkbook with him. Therefore Watson did not wish to invest money with Thurston.	P : Watson wished to invest money with Thurston. Q : Watson had his checkbook with him. If P then Q . $\sim Q$ $\therefore \sim P$
If \overline{AC} is the longest side in $\triangle ABC$, then $\angle B$ is the largest angle in $\triangle ABC$. $\angle B$ is not the largest angle in $\triangle ABC$. Therefore \overline{AC} is not the longest side in $\triangle ABC$.	P : \overline{AC} is the longest side in $\triangle ABC$. Q : $\angle B$ is the largest angle in $\triangle ABC$. If P then Q . $\sim Q$ $\therefore \sim P$

Activity

It's Elementary!

In this activity you'll apply what you have learned about *Modus Ponens* (MP) and *Modus Tollens* (MT). You'll also get practice using the symbols of logic such as P and $\sim P$ as statements and \therefore for “so” or “therefore.” To shorten your work even further, you can symbolize the conditional “If P then Q ” as $P \rightarrow Q$. Then *Modus Ponens* and *Modus Tollens* written symbolically look like this:

Modus Ponens	Modus Tollens
$P \rightarrow Q$	$R \rightarrow S$
P	$\sim S$
$\therefore Q$	$\therefore \sim R$

- Step 1 Use logic symbols to translate parts a–e. Tell whether *Modus Ponens* or *Modus Tollens* is used to make the reasoning valid.
- If Watson was playing billiards, then he was playing with Thurston. Watson was playing billiards. Therefore Watson was playing with Thurston.
 - Every cheerleader at Washington High School is in the 11th grade. Mark is a cheerleader at Washington High School. Therefore Mark is in the 11th grade.
 - If Carolyn studies, then she does well on tests. Carolyn did not do well on her tests, so she must not have studied.
 - If \overline{ED} is a midsegment in $\triangle ABC$, then \overline{ED} is parallel to a side of $\triangle ABC$. \overline{ED} is a midsegment in $\triangle ABC$. Therefore \overline{ED} is parallel to a side of $\triangle ABC$.
 - If \overline{ED} is a midsegment in $\triangle ABC$, then \overline{ED} is parallel to a side of $\triangle ABC$. \overline{ED} is not parallel to a side of $\triangle ABC$. Therefore \overline{ED} is not a midsegment in $\triangle ABC$.
- Step 2 Use logic symbols to translate parts a–e. If the two premises fit the valid reasoning pattern of *Modus Ponens* or *Modus Tollens*, state the conclusion symbolically and translate it into English. Tell whether *Modus Ponens* or *Modus Tollens* is used to make the reasoning valid. Otherwise write “no valid conclusion.”
- If Aurora passes her Spanish test, then she will graduate. Aurora passes the test.
 - The diagonals of $ABCD$ are not congruent. If $ABCD$ is a rectangle, then its diagonals are congruent.
 - If yesterday was Thursday, then there is no school tomorrow. There is no school tomorrow.
 - If you don’t use Shining Smile toothpaste, then you won’t be successful. You do not use Shining Smile toothpaste.
 - If squiggles are flitz, then ruggles are bodrum. Ruggles are not bodrum.
- Step 3 Identify each symbolic argument as *Modus Ponens* or *Modus Tollens*. If the argument is not valid, write “no valid conclusion.”
- | | | |
|---|---|---|
| <p>a. $P \rightarrow S$
 P
 $\therefore S$</p> | <p>b. $\sim T \rightarrow P$
 $\sim T$
 $\therefore P$</p> | <p>c. $R \rightarrow \sim Q$
 Q
 $\therefore \sim R$</p> |
| <p>d. $Q \rightarrow S$
 S
 $\therefore \sim Q$</p> | <p>e. $Q \rightarrow P$
 $\sim Q$
 $\therefore \sim P$</p> | <p>f. $\sim R \rightarrow S$
 $\sim S$
 $\therefore R$</p> |
| <p>g. $\sim P \rightarrow (R \rightarrow Q)$
 $\sim P$
 $\therefore (R \rightarrow Q)$</p> | <p>h. $(T \rightarrow \sim P) \rightarrow Q$
 $\sim Q$
 $\therefore \sim (T \rightarrow \sim P)$</p> | <p>i. $P \rightarrow (\sim R \rightarrow P)$
 $(\sim R \rightarrow P)$
 $\therefore P$</p> |

CHAPTER 10 REVIEW

In this chapter you discovered a number of formulas for finding volumes. It's as important to remember how you discovered these formulas as it is to remember the formulas themselves. For example, if you recall pouring the contents of a cone into a cylinder with the same base and height, you may recall that the volume of the cone is one-third the volume of the cylinder. Making connections will help too. Recall that prisms and cylinders share the same volume formula because their shapes—two congruent bases connected by lateral faces—are alike.



You should also be able to find the surface area of a sphere. The formula for the surface area of a sphere was intentionally not included in Chapter 8, where you first learned about surface area. Look back at the investigations in Lesson 8.7 and explain why the surface area formula requires that you know volume.

As you have seen, volume formulas can be applied to many practical problems. Volume also has many extensions such as calculating displacement and density.

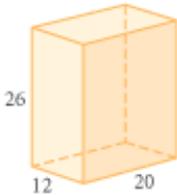


EXERCISES

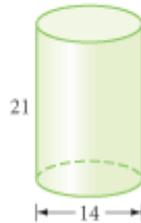
1. How are a prism and a cylinder alike?
2. What does a cone have in common with a pyramid?

For Exercises 3–8, find the volume of each solid. Each quadrilateral is a rectangle. All solids are right (not oblique). All measurements are in centimeters.

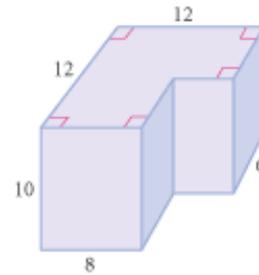
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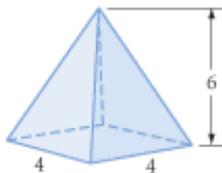
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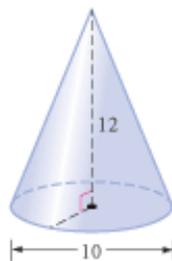
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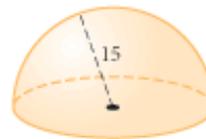
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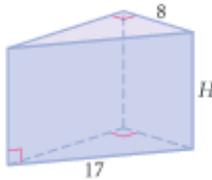


8.

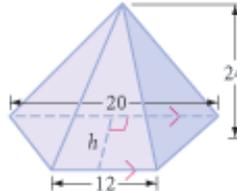


For Exercises 9–12, calculate each unknown length given the volume of the solid. All measurements are in centimeters.

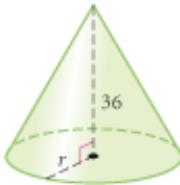
9. Find $H.V = 768 \text{ cm}^3$



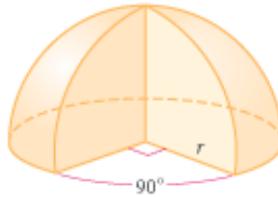
10. Find $h.V = 896 \text{ cm}^3$



11. Find $r.V = 1728\pi \text{ cm}^3$



12. Find $r.V = 256\pi \text{ cm}^3$



13. Find the volume of a rectangular prism whose dimensions are twice those of another rectangular prism that has a volume of 120 cm^3 .
14. Find the height of a cone with a volume of 138π cubic meters and a base area of 46π square meters.
15. Find the volume of a regular hexagonal prism that has a cylinder drilled from its center. Each side of the hexagonal base measures 8 cm. The height of the prism is 16 cm. The cylinder has a radius of 6 cm. Express your answer to the nearest cubic centimeter.
16. Two rectangular prisms have equal heights but unequal bases. Each dimension of the smaller solid's base is half each dimension of the larger solid's base. The volume of the larger solid is how many times as great as the volume of the smaller solid?
17. The "extra large" popcorn container is a right rectangular prism with dimensions 3 in. by 3 in. by 6 in. The "jumbo" is a cone with height 12 in. and diameter 8 in. The "colossal" is a right cylinder with diameter 10 in. and height 10 in.
 - a. Find the volume of all three containers.
 - b. The volume of the "colossal" is approximately how many times as great as that of the "extra large"?
18. Two solid cylinders are made of the same material. Cylinder A is six times as tall as cylinder B, but the diameter of cylinder B is four times the diameter of cylinder A. Which cylinder weighs more? How many times as much?

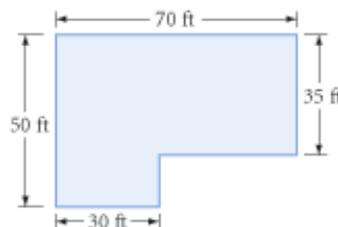


PIRANHA CLUB by B. Grace. Reprinted with special permission of King Features Syndicate.

19. **Application** Rosa Avila is a plumbing contractor. She needs to deliver 200 lengths of steel pipe to a construction site. Each cylindrical steel pipe is 160 cm long, has an outer diameter of 6 cm, and has an inner diameter of 5 cm. Rosa needs to know whether her quarter-tonne truck can handle the weight of the pipes. To the nearest kilogram, what is the mass of these 200 pipes? How many loads will Rosa have to transport to deliver the 200 lengths of steel pipe? (Steel has a density of about 7.7 g/cm^3 . One tonne equals 1000 kg.)

20. A ball is placed snugly into the smallest possible box that will completely contain the ball. What percentage of the box is filled by the ball?

21. **Application** The blueprint for a cement slab floor is shown at right. How many cubic yards of cement are needed for ten identical floors that are each 4 inches thick?



22. **Application** A prep chef has just made two dozen meatballs. Each meatball has a 2-inch diameter. Right now, before the meatballs are added, the sauce is 2 inches from the top of the 14-inch-diameter pot. Will the sauce spill over when the chef adds the meatballs to the pot?



23. To solve a crime, Betty Holmes, who claims to be Sherlock's distant cousin, and her friend Professor Hilton Gardens must determine the density of a metal art deco statue with mass 5560 g. She places it into a graduated glass prism filled with water and finds that the level rises 4 cm. Each edge of the glass prism's regular hexagonal base measures 5 cm. Professor Gardens calculates the statue's volume, then its density. Next, Betty Holmes checks the density table (see page 551) to determine if the statue is platinum. If so, it is the missing piece from her client's collection and Inspector Clouseau is the thief. If not, then the Baron is guilty of fraud. What is the statue made of?

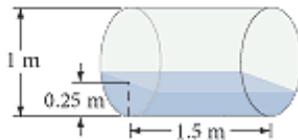
24. Can you pick up a solid steel ball of radius 6 inches? Steel has a density of 0.28 pound per cubic inch. To the nearest pound, what is the weight of the ball?

25. To the nearest pound, what is the weight of a hollow steel ball with an outer diameter of 14 inches and a thickness of 2 inches?

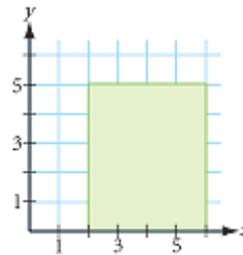
26. A hollow steel ball has a diameter of 14 inches and weighs 327.36 pounds. Find the thickness of the ball.



27. A water barrel that is 1 m in diameter and 1.5 m long is partially filled. By tapping on its sides, you estimate that the water is 0.25 m deep at the deepest point. What is the volume of the water in cubic meters?

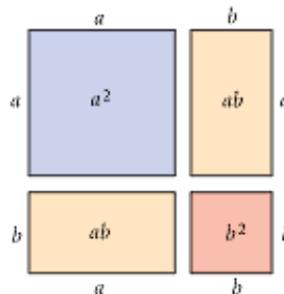
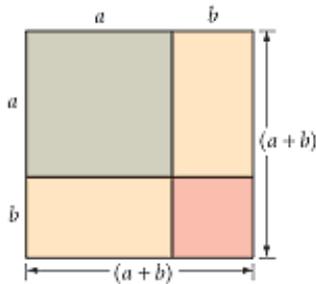


28. Find the volume of the solid formed by rotating the shaded figure about the y -axis.

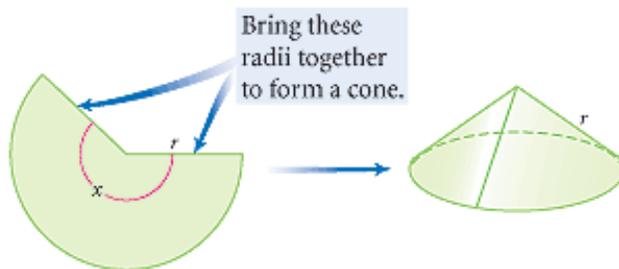


TAKE ANOTHER LOOK

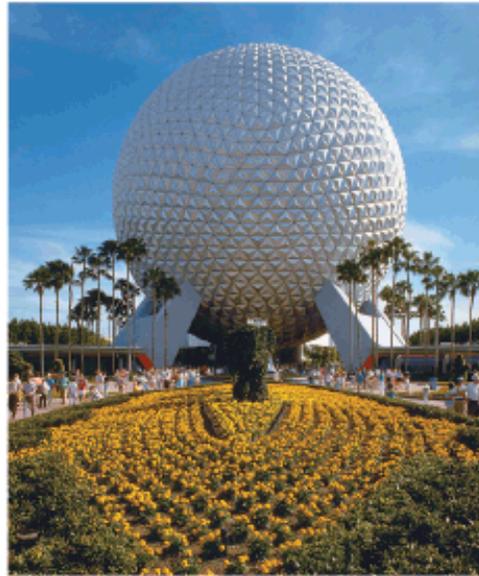
1. You may be familiar with the area model of the expression $(a + b)^2$, shown below. Draw or build a volume model of the expression $(a + b)^3$. How many distinct pieces does your model have? What's the volume of each type of piece? Use your model to write the expression $(a + b)^3$ in expanded form.



2. Use algebra to show that if you double all three dimensions of a prism, a cylinder, a pyramid, or a cone, the volume is increased eightfold, but the surface area is increased only four times.
3. Any sector of a circle can be rolled into a cone. Find a way to calculate the volume of a cone given the radius and central angle of the sector.



4. Build a model of three pyramids with equal volumes that you can assemble into a prism. It's easier to start with the prism and then separate it into the pyramids.
5. Derive the Sphere Volume Conjecture by using a pair of hollow shapes different from those you used in the Investigation The Formula for the Volume of a Sphere. Or use two solids made of the same material and compare weights. Explain what you did and how it demonstrates the conjecture.
6. Spaceship Earth, located at the Epcot center in Orlando, Florida, is made of polygonal regions arranged in little pyramids. The building appears spherical, but the surface is not smooth. If a perfectly smooth sphere had the same volume as Spaceship Earth, would it have the same surface area? If not, which would be greater, the surface area of the smooth sphere or of the bumpy sphere? Explain.



Spaceship Earth, the Epcot center's signature structure, opened in 1982 in Walt Disney World. It features a ride that chronicles the history of communication technology.

Assessing What You've Learned



UPDATE YOUR PORTFOLIO Choose a project, a Take Another Look activity, or one of the more challenging problems or puzzles you did in this chapter to add to your portfolio.



WRITE IN YOUR JOURNAL Describe your own problem-solving approach. Are there certain steps you follow when you solve a challenging problem? What are some of your most successful problem-solving strategies?



ORGANIZE YOUR NOTEBOOK Review your notebook to be sure it's complete and well organized. Be sure you have all the conjectures in your conjecture list. Write a one-page summary of Chapter 10.



PERFORMANCE ASSESSMENT While a classmate, friend, family member, or teacher observes, demonstrate how to derive one or more of the volume formulas. Explain what you're doing at each step.



WRITE TEST ITEMS Work with classmates to write test items for this chapter. Include simple exercises and complex application problems. Try to demonstrate more than one approach in your solutions.



GIVE A PRESENTATION Give a presentation about one or more of the volume conjectures. Use posters, models, or visual aids to support your presentation.